Breakdown of self-similarity at the crests of large amplitude standing water waves, and other applications of overdetermined shooting methods

Frontiers in Computation: New Methods for Complex Mechanics, Advanced Materials, Interfaces, and Stochastics

Jon Wilkening
UC Berkeley and LBNL
970 Evans Hall, Berkeley, CA 94720-3840

Abstract

Two-point boundary value problems governed by nonlinear partial differential equations arise frequently in applied mathematics. When the underlying system is infinite dimensional, the equations must be discretized before solving them numerically. Truncation errors lead to loss of accuracy in the highest frequency modes of the numerical solution, which can cause difficulty for the convergence of shooting methods. We have developed a collection of techniques in which robustness is achieved in these problems by posing them as overdetermined nonlinear systems. We solve these systems using a combination of adjoint-based optimal control techniques and trust region methods. For the latter, the Jacobian is computed rapidly by solving the linearized equations in parallel, using a GPU.

Using this approach, we are able to answer a long-standing open question, posed by Penney and Price in 1952, on the limiting behavior at the crests of large amplitude standing water waves. By analogy with Stokes' conjecture that the largest amplitude traveling wave would feature sharp, 120 degree interior crest angles, Penney and Price predicted that the largest amplitude standing wave would develop sharp 90 degree corner angles each time the fluid comes to rest. Due to the difficulty of maintaining accuracy in nearly singular free surface flow calculations, previous numerical studies have reached different conclusions regarding the form of the largest amplitude standing wave. Mercer and Roberts predicted interior crest angles as sharp as 60 degree; Schultz et. al. predicted the formation of a cusp rather than a corner; and Okamura found that Penney and Price's conjecture of a 90 degree corner holds true. Our work shows that these discrepancies are due to lack of sufficient spatial resolution. Refined calculations reveal that the Penney and Price conjecture is false due to a breakdown of self-similarity at the crests of very large amplitude standing waves. As one progresses into this large amplitude regime, standing waves develop increasingly complex behavior at small scales. Thus, the assumption that a "largest amplitude" standing wave exists, terminating the bifurcation curve through the formation of a geometric singularity, appears to be incorrect.

To reach these conclusions, it was essential to develop a highly accurate, large scale water wave code. For the Dirichlet-Neumann operator, we used a spectrally accurate boundary integral collocation method in 2d, and a 5th order multigrid finite element method in 3d. The 2d calculations were performed in double and quadruple precision using a 448-core GPU. For time-stepping, we used an 8th order Runge-Kutta method for the double-precision calculations, and a 15th order spectral deferred correction method for the quadruple-precision calculations. Many of the 3d calculations were performed on the Lawrencium cluster at LBNL.

If time permits, I will also discuss our current work using overdetermined shooting methods to study resonant modes in rolling tires, acoustic waves in compressible gas dynamics, mixing in viscoelastic flows, and stability transitions in mode-locked lasers.