A framework to analyze regime behaviors and metastability in the climate system

Eric Vanden-Eijnden
Courant Institute of Mathematical Sciences
New York University

In collaboration with Weiqing Ren (NUS) and Xu Yang (NYU)

Part of DOE funded project with Guang Lin (PNNL)

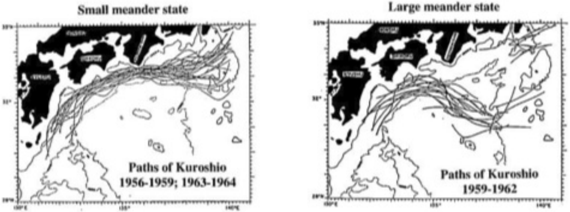
Outline

Regimes in Climate - examples

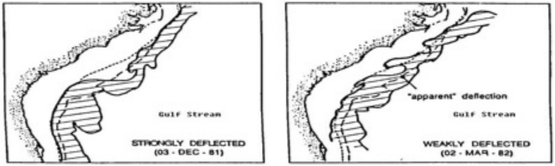
Motivation
Methodology
Stochastic ODE model
Stochastic PDE model
Conclusions

Weather pattern over central Europe (Baur, 1951), North Atlantic Oscillation (NAO), Arctic Oscillation (AO), Pacific-North American (PNA) pattern, etc.

Kuroshio current (Taft, 1972)



Gulf stream (Bane and Dewar, 1988)



Regimes in Climate

- Regimes as multiple equilibrium states of barotropic models (Charney–Devore 79, Legras–Ghil 85, Tung–Tosenthal 85, ...).
- Regimes as most likely states of equilibrium distribution from statistical mechanics (Miller 90, Robert–Sommeria 91, Majda *et al.* 05, 06, ...).
- Data-driven approaches (Majda *et al.* 10, 11, ...).

Dynamics of transitions between regimes?

- Transitions caused by effect of noise - typically analyzed in small dimensional idealized models (Hasselmann 76, Egger 81, Saravanan *et al.*, Sura 02, ...).
- Transition as heteroclinic orbits (Crommelin *et al* 03, Selten–Branstator 04, Ide–Ghil 04, ...).

No dynamical information from standard equilibrium statistical mechanics approaches!

Our strategy

- Use equilibrium statistical mechanics framework (information theory) to determine invariant measure (IM) of the climate system;
- Establish conditions under which this IM is multimodal, i.e. may displays several regimes;
- *Re-install dynamics* - add appropriate forcing and damping terms in the bare equations to guarantee that the resulting stochastic partial differential equations (SPDEs) have the correct IM;
- Analyze these SPDEs in the small noise limit using large deviation theory and the string method to find most likely pathways between the different regimes.

Outline

Two-dimensional barotropic QG equations

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q + U(t)\frac{\partial q}{\partial x} + \beta\frac{\partial \psi}{\partial x} = \mathcal{D}_q(\Delta)q + F_q,$$
$$q = \Delta\psi + h,$$
$$\frac{\mathrm{d}U}{\mathrm{d}t} = \fint h\frac{\partial \psi}{\partial x}\,\mathrm{d}x\,\mathrm{d}y + \mathcal{D}_U(U) + F_U,$$

Here $\psi(t,x,y)$ is the stream function, $q(t,x,y)$ the vorticity, $h(x,y)$ the bottom topography, $U(t)$ the zonal base flow and β the beta-plane parameter (Pedlosky 79, Vallis 05).

Inviscid equations with $\mathcal{D} = F = 0$ preserve

Energy $\quad E = \frac{1}{2}U^2 + \frac{1}{2}\fint |\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}y,$

Enstrophy $\quad Q = \beta U + \frac{1}{2}\fint |q|^2\,\mathrm{d}x\,\mathrm{d}y.$

Predictions of equilibrium statistical mechanics

IM maximizes the entropy $-\int \mu \log \mu$ under the constraints:

Canonical: $\displaystyle\int Q \, d\mu = Q_0$, Microcanonical: $E = E_0$.

$$d\mu = Z^{-1} \exp(-\varepsilon^{-1} Q) \delta(E - E_0) \, dU d\psi,$$

- Most likely states = selective decay states – in QG flow, enstrophy varies faster than energy.
- Only choice leading to multimodal distribution.

Dynamics? choose $\mathcal{D}$ and F so that we formally have the invariant measure (cf strategy of Landau-Lifshitz for NS equations)

Outline

Single-mode truncation

$$\psi(t,x,y) = a(t)\sin(kx) + b(t)\cos(kx),$$
$$h(x,y) = H\sin(kx), \quad (k \neq 0).$$

Inviscid equations reduce to:

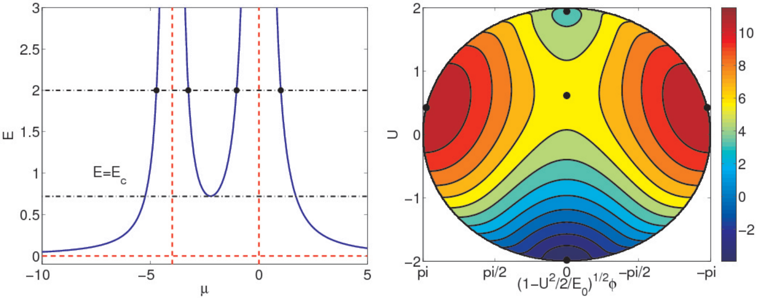
$$\dot{a} = kb(t)\left(U(t) - \frac{\beta}{k^2}\right),$$
$$\dot{b} = -ka(t)\left(U(t) - \frac{\beta}{k^2}\right) + \frac{U(t)H}{k},$$
$$\dot{U} = -\frac{b(t)Hk}{2}.$$

The energy and enstrophy now read:

$$E = \frac{1}{2}U^2 + \frac{k^2}{4}(a^2 + b^2),$$
$$Q = \beta U + \frac{1}{4}\left((H - k^2 a)^2 + k^4 b^2\right).$$

Invariant measure $\sim \exp(-\varepsilon^{-1} Q)\delta(E - E_0)$

Enstrophy landscape on energy surface $E_0 = 2$:



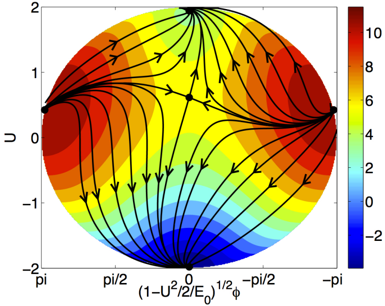
Most likely states are given by the minimizers of Q subject to the constraint $E = E_0$.

Reinstall dynamics

$$\dot{\boldsymbol{X}} = \boldsymbol{B}(\boldsymbol{X}) - \mathcal{P}(\boldsymbol{X}) M_\gamma \nabla_{\boldsymbol{X}} Q(\boldsymbol{X}) + \sqrt{2\varepsilon}\,\mathcal{P}(\boldsymbol{X}) M_\gamma^{1/2} \circ \eta,$$

with the projection operator

$$\mathcal{P} = \mathrm{Id} - \frac{M_\gamma \nabla_{\boldsymbol{X}} E \otimes \nabla_{\boldsymbol{X}} E}{\langle \nabla_{\boldsymbol{X}} E, M_\gamma \nabla_{\boldsymbol{X}} E \rangle}.$$



Deterministic flow

Maximum likelihood path

In the small noise limit, transition occurs via maximum likelihood path (MLP) of large deviation theory.

MLP is minimizer in

$$\inf_{\mathcal{M}} \frac{1}{2} \int_{-T}^{T} \left| \left(\mathcal{P}(\varphi) M_\gamma\right)^{-1/2} \left(\dot{\varphi} - \mathcal{P}(\varphi)\left(\boldsymbol{B}(\varphi) - M_\gamma \nabla_{\boldsymbol{X}} Q(\varphi)\right)\right) \right|^2 \mathrm{d}t,$$

where

$$\mathcal{M} = \left\{ \varphi \middle| \varphi(t) \in \mathrm{C}([-T, T], \mathbb{R}^3),\ \dot{\varphi} \perp \nabla_{\boldsymbol{X}} E,\ \varphi(-T) = \boldsymbol{X}_0,\ \varphi(T) = \boldsymbol{X}_1 \right\}$$

Nongradient system with conservative nongradient force

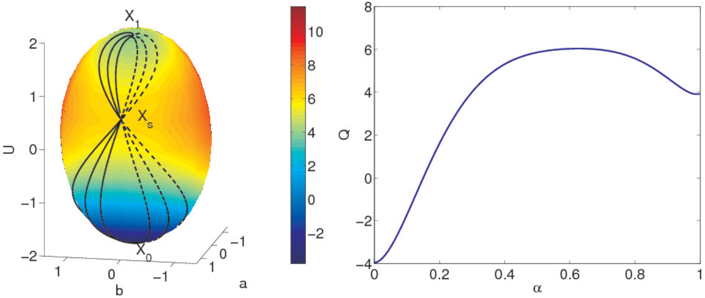
$$\nabla_{\boldsymbol{X}} E \perp \boldsymbol{B}, \quad \nabla_{\boldsymbol{X}} Q \perp \boldsymbol{B}, \quad \mathcal{P}\boldsymbol{B} = \boldsymbol{B}.$$

MLP solves

$$\begin{cases} 0 = \left[\boldsymbol{B} + \mathcal{P}\left(M_\gamma \nabla_{\boldsymbol{X}} Q(\varphi)\right)\right]^\perp & \text{(uphill)} \\ 0 = \left[\boldsymbol{B} - \mathcal{P}\left(M_\gamma \nabla_{\boldsymbol{X}} Q(\varphi)\right)\right]^\perp & \text{(downhill)} \end{cases}$$

where $[\cdot]^\perp$ denotes projection perpendicular to the path.

Computed by string method



Here $\beta = k = 2$, $H = 1$, $E_0 = 2$ and $M_\gamma = 0.5, 1, 2$.

Outline

Metastable states of invariant measure

Minimize the enstrophy Q subject to the constraint $E = E_0$:

$$\min_{U,\psi} \ Q = \beta U + \frac{1}{2} \fint q^2 \, \mathrm{d}x \, \mathrm{d}y,$$
$$\text{s.t.} \ \frac{1}{2} U^2 + \frac{1}{2} \fint |\nabla \psi|^2 \, \mathrm{d}x \, \mathrm{d}y = E_0.$$

Using Lagrange multiplier,

$$\mu U_c = -\beta, \quad \mu \Delta \psi_c = \Delta(\Delta \psi_c + h).$$

N.B. These are the stationary solutions of the original system without dissipation and forcing.

Number of critical points $\Rightarrow$ roots of μ:

$$E_0 = \frac{\beta^2}{2\mu^2} + \frac{1}{2}\sum_{\boldsymbol{k}} \frac{|\boldsymbol{k}|^2 \left|\widehat{h}_{\boldsymbol{k}}\right|^2}{\left(\mu + |\boldsymbol{k}|^2\right)^2},$$

Stability $\Rightarrow$ signs of eigenvalues of Hessian matrix:

$$(M_Q)_{\boldsymbol{k}\boldsymbol{k}'} = \begin{cases} \dfrac{\mu^3 |\boldsymbol{k}|^4 \left|\widehat{h}_{\boldsymbol{k}}\right|^2}{\beta^2 \left(\mu + |\boldsymbol{k}|^2\right)^2} - |\mu| \, |\boldsymbol{k}|^2 + |\boldsymbol{k}|^4, & \text{if} \quad \boldsymbol{k} = \boldsymbol{k}', \\ \dfrac{\mu^3 |\boldsymbol{k}|^2 |\boldsymbol{k}'|^2 \widehat{h}_{\boldsymbol{k}} \widehat{h}^*_{\boldsymbol{k}'}}{\beta^2 \left(\mu + |\boldsymbol{k}|^2\right)\left(\mu + |\boldsymbol{k}'|^2\right)}, & \text{if} \quad \boldsymbol{k} \neq \boldsymbol{k}'. \end{cases}$$

Stochastic partial differential equations

$$\frac{\partial \psi}{\partial t} = -\Delta^{-1}\left(\nabla^{\perp}\psi \cdot \nabla q + U(t)\frac{\partial q}{\partial x} + \beta\frac{\partial \psi}{\partial x}\right)$$
$$- \gamma_1 \Delta(\Delta\psi + h) + \sqrt{2\varepsilon\gamma_1}\eta(t,x,y) + \gamma_1 \frac{\delta E}{\delta \psi}\lambda(t),$$
$$\frac{\mathrm{d}U}{\mathrm{d}t} = \fint h\frac{\partial \psi}{\partial x}\,\mathrm{d}\boldsymbol{x} - \gamma_2\beta + \sqrt{2\varepsilon\gamma_2}\xi(t) + \gamma_2\frac{\delta E}{\delta U}\lambda(t),$$

where $q = \Delta\psi + h$ and the Lagrange multiplier $\lambda(t)$ is

$$\lambda(t) = \frac{\fint (q-h)(-\gamma_1\Delta q + \sqrt{2\varepsilon\gamma_1}\circ\eta)\,\mathrm{d}\boldsymbol{x} + U(\gamma_2\beta - \sqrt{2\varepsilon\gamma_2}\circ\xi)}{\fint (q-h)\gamma_1(q-h)\,\mathrm{d}\boldsymbol{x} + \gamma_2 U^2}.$$

N.B. γ_1 and γ_2 can be pseudo-differential operators.

Dissipation via Ekman drag: $\gamma_1 = \gamma\Delta^{-2}$, $\gamma_2 = \widetilde{\gamma}$

$$
\begin{aligned}
\frac{\partial q}{\partial t} + \nabla^{\perp}\psi\cdot\nabla q + U(t)\frac{\partial q}{\partial x} + \beta\frac{\partial \psi}{\partial x} &= -\gamma q + \sqrt{2\varepsilon\gamma}\,\eta(t,x,y) - \gamma\psi\lambda(t),\\
q &= \Delta\psi + h,\\
\frac{\mathrm{d}U}{\mathrm{d}t} &= \fint h\frac{\partial \psi}{\partial x}\,\mathrm{d}\boldsymbol{x} - \widetilde{\gamma}\beta + \sqrt{2\varepsilon\widetilde{\gamma}}\,\xi(t) + \widetilde{\gamma}U\lambda(t),
\end{aligned}
$$

where

$$
\lambda(t) = \frac{\fint \psi(-\gamma q + \sqrt{2\varepsilon\gamma}\circ\eta)\,\mathrm{d}\boldsymbol{x} + U(\widetilde{\gamma}\beta - \sqrt{2\varepsilon\widetilde{\gamma}}\circ\xi)}{\gamma\fint|\psi|^2\,\mathrm{d}\boldsymbol{x} + \widetilde{\gamma}U^2}.
$$

N.B. fixing the damping fixes the forcing by fluctuation-dissipation theorem (FDT).

String method to compute MLP

1. Evolve the string according to

$$\tilde{\varphi} = \varphi^n + \delta t(\pm \boldsymbol{B}^n + \boldsymbol{G}^n),$$

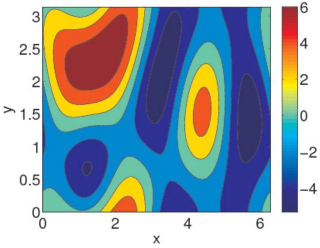
where $\varphi = (q, U)^T$, and

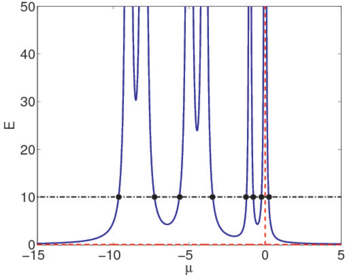
$$\boldsymbol{B} = \begin{pmatrix} -\nabla^\perp \psi \cdot \nabla q - U \frac{\partial q}{\partial x} - \beta \frac{\partial \psi}{\partial x} \\ \fint h \dfrac{\partial \psi}{\partial x} \, \mathrm{d}\boldsymbol{x} \end{pmatrix}, \quad \boldsymbol{G} = -\begin{pmatrix} \gamma q \\ \tilde{\gamma} \beta \end{pmatrix}.$$

2. Reparametrize $\tilde{\varphi}$ via arc-length, to obtain $\bar{\varphi}$.
3. Project $\bar{\varphi}$ on the energy surface $E = E_0$, to obtain φ^{n+1}.

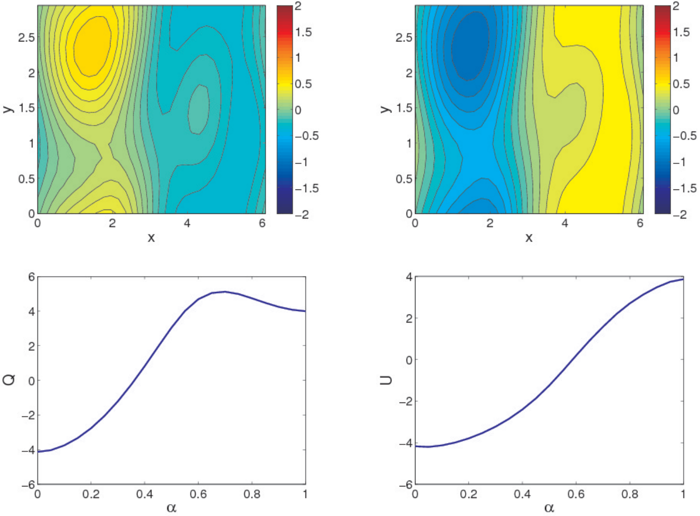
Example (16-mode topography)

We take the domain size as $L_x = 2\pi,\ L_y = \pi$. The parameters are chosen as $\beta = 1,\ E_0 = 10$. The friction coefficients are taken as $\gamma = \widetilde{\gamma} = 4$.

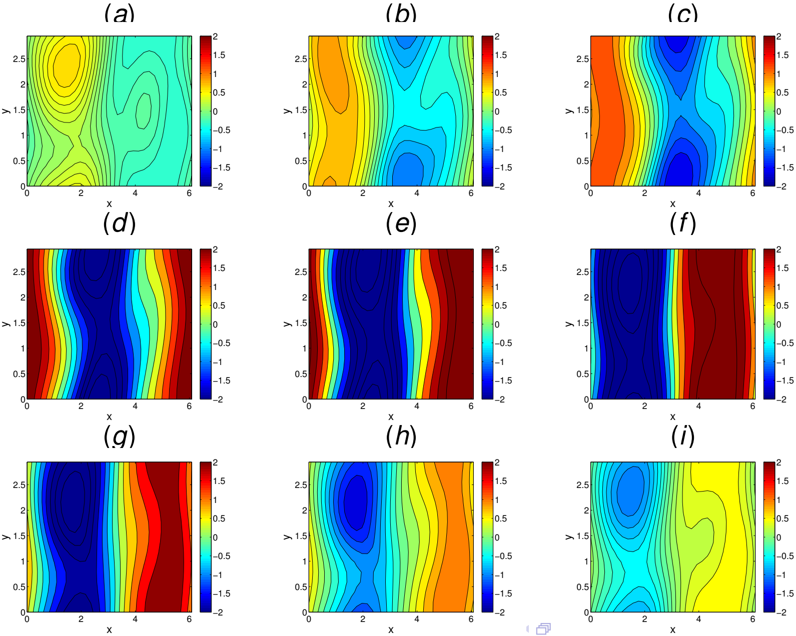




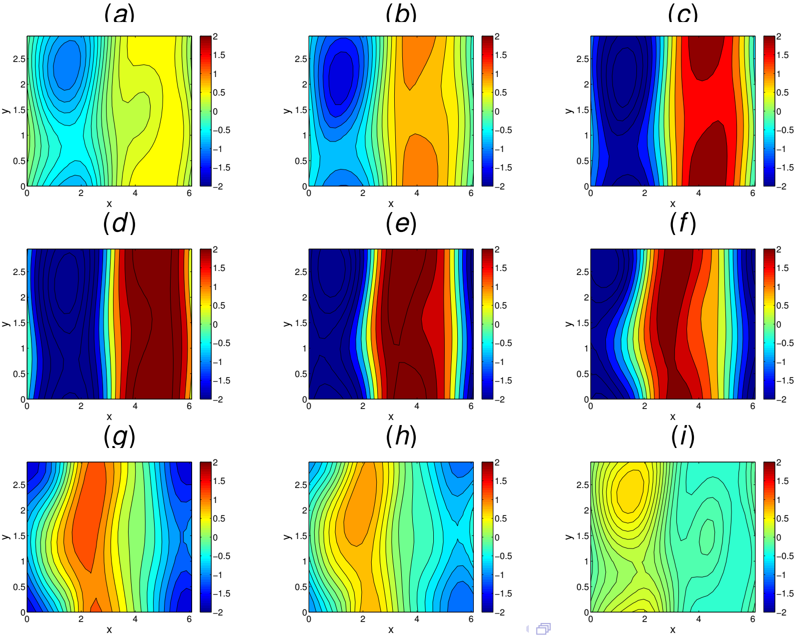
Metastable vorticity patterns and the pathways



Dynamics of streamlines and vorticity



Dynamics of streamlines and vorticity



Outline

Conclusion:

- Design SPDE models that are consitent with predictions of equilibrium statistical mechanics and can capture the noise-induced transition between different selective decay states (representing different climate regimes).
- Calculate the maximum likelihood path of large deviation theory by the string method to describe the transitions between different regimes in the small noise limit.
- Can be generalized to other models with different forcing and dissipation, including cases where statistical mechanics framework breaks down (nonequilibrium statistical steady states).
- Can be generalized to situations with weak damping and forcing (different order of limits than weak noise).