AMG Transfers for Nonsymmetric Systems:
Schur Complements & Galerkin Projections

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Goal

- Mathematical understanding & framework for developing a new flexible algebraic multigrid platform
  - Classes of transfer algorithm strategies including for non-symmetric
  - arbitrary coarsening, flexible coarse basis function support, accurate interpolation of arbitrary important modes

Outline

- Schur complement / AMG connection
- Schur complements & Galerkin projections
- Choosing subspaces …
- Iterative solution of Galerkin system
- Practical aspects & numerical demonstration

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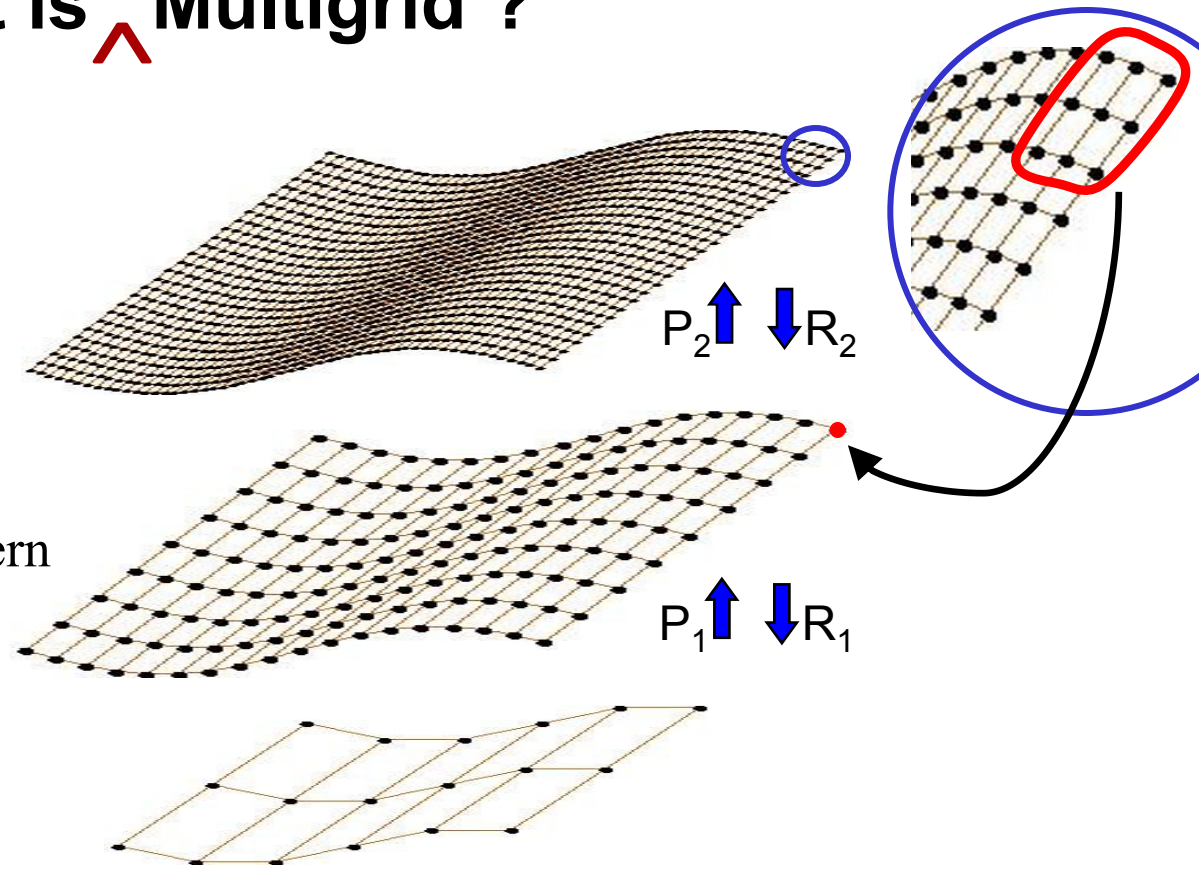
What is Algebraic Multigrid ?

Solve $A_3 u_3 = f_3$

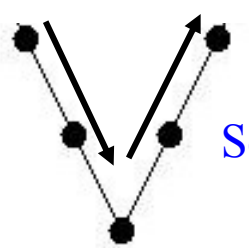
- Construct Graph & Coarsen
- Determine P_i & R_i sparsity pattern
- Determine P_i & R_i's coefs
- Project: $A_i = R_i A_{i+1} P_i$



Smooth $A_3 u_3 = f_3$. Set $f_2 = R_2 r_3$.

Smooth $A_2 u_2 = f_2$. Set $f_1 = R_1 r_2$.

Solve $A_1 u_1 = f_1$ directly.



Set $u_3 = u_3 + P_2 u_2$. Smooth $A_3 u_3 = f_3$.

Set $u_2 = u_2 + P_1 u_1$. Smooth $A_2 u_2 = f_2$.

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Idealized AMG

$$\begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix}^{-1} = \underbrace{\begin{pmatrix} I & -A_{ff}^{-1}A_{fc} \\ & I \end{pmatrix}}_{P_{opt}} \begin{pmatrix} A_{ff}^{-1} & \\ & S^{-1} \end{pmatrix} \underbrace{\begin{pmatrix} I & \\ -A_{cf}A_{ff}^{-1} & I \end{pmatrix}}_{R_{opt}}$$

$$S = A_{cc} - A_{cf}A_{ff}^{-1}A_{fc}$$

Theorem

Let

$$M^{-1} = \underbrace{\begin{pmatrix} I & P_f \\ & I \end{pmatrix}}_{P} \begin{pmatrix} A_{ff}^{-1} & \\ & A_{l+1}^{-1} \end{pmatrix} \underbrace{\begin{pmatrix} I & \\ R_f & I \end{pmatrix}}_{R}$$

Then

$$\rho(M^{-1}A) = \left\{1, 1 \pm \sqrt{1 - \rho(A_{l+1}^{-1}S)}\right\}$$

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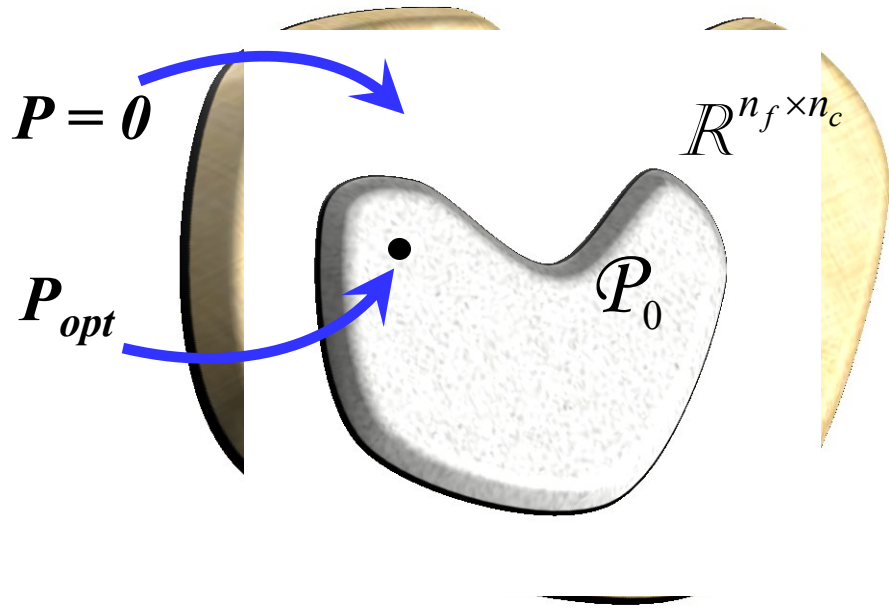
Galerkin $\Rightarrow$ Schur Complement

Consider

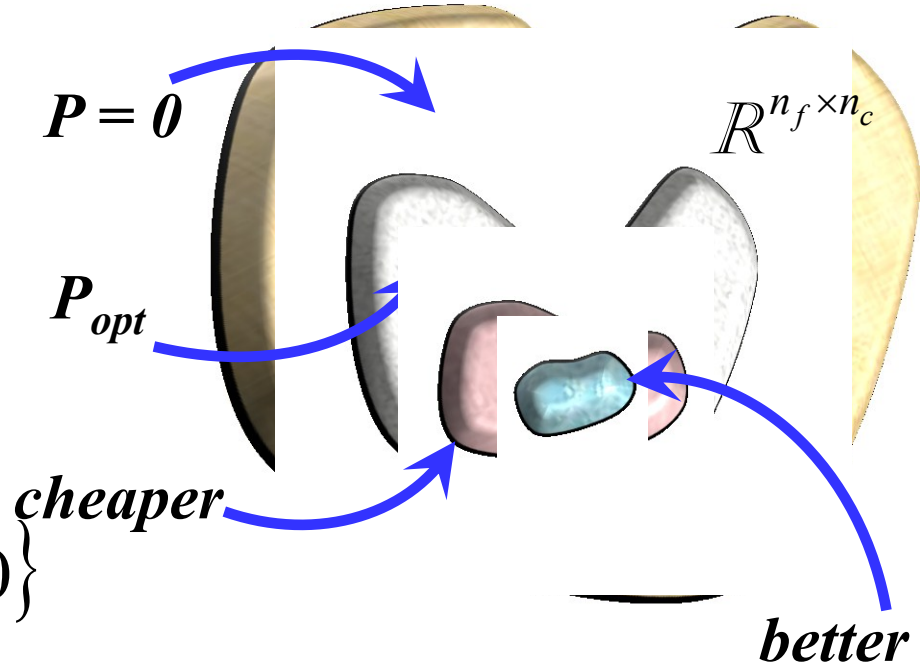
$$AP = 0 \qquad \& \qquad RA = 0$$

Solution $P = 0, R = 0$!



Subspaces

Assume we are given sparsity pattern N_f .



Define

$$\mathcal{Z} = \left\{ P \in \mathcal{P} : N_{ij} = 0 \;\Rightarrow\; P_{ij} = 0 \right\}$$

where

$$N = \begin{pmatrix} N_f \\ I \end{pmatrix} .$$

Then

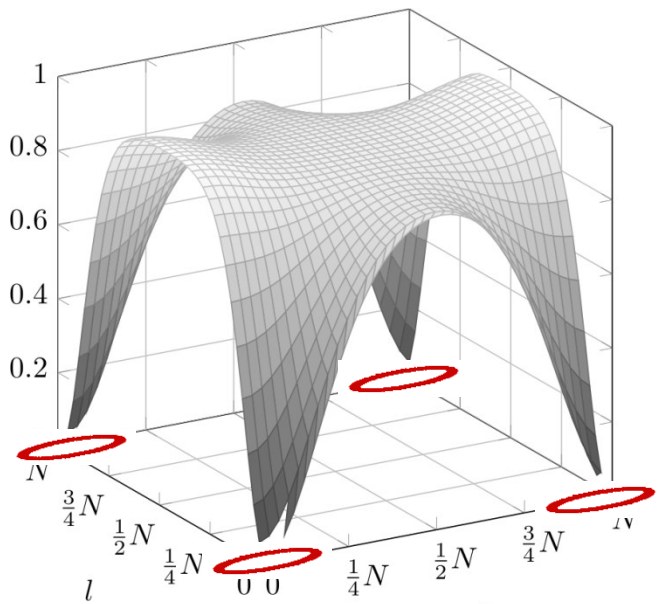
$$AP \perp \mathcal{Z} \quad \text{with} \quad P \in \mathcal{Z}_0 \Rightarrow \textit{truncated Schur complement}$$

Cheaper … but not yet better

Recall $$\rho(M^{-1}A) = \left\{1, 1 \pm \sqrt{1 - \rho(A_{l+1}^{-1} S)}\right\}$$

and consider discretization of $u_{xx} + u_{yy} + .8\, u_y$.

Using uniform f/c splitting and standard N_f ,then in Fourier space $\rho\,(M^{-1}\,A)$ is



want

$A_{l+1}\, v = S\, v$

for constant v

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Near Null Space Preservation

Define

$$\mathcal{Z}_0 = \left\{ P \in \mathcal{P}_0 : N_{ij} = 0 \;\Rightarrow\; P_{ij} = 0 \quad and \quad PB_{l+1} = B \right\}$$

$$\mathcal{Z} = \left\{ P \in \mathcal{P} : N_{ij} = 0 \;\Rightarrow\; P_{ij} = 0 \quad and \quad PB_{l+1} = 0 \right\}$$

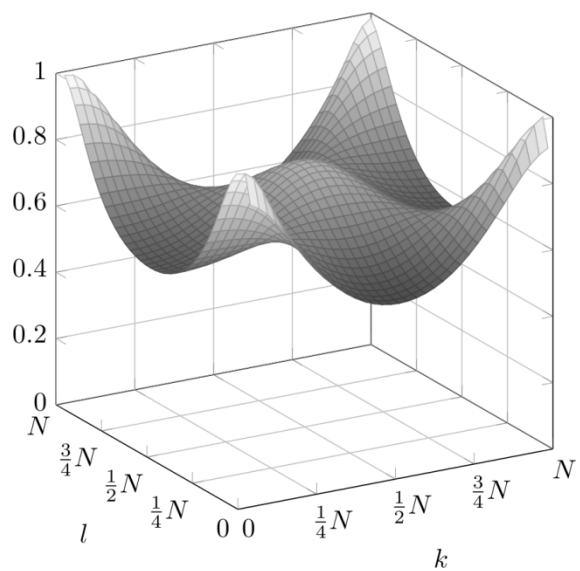
where

$$A\,B \approx 0 \qquad \text{and} \quad B_{l+1} \text{ injected from } B$$

Then

$$AP \perp \mathcal{Z} \quad \text{with} \quad P \in \mathcal{Z}_0$$

for same problem as before $\Rightarrow$



Solving the Galerkin System

Define $\breve{A} = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}$ $\widehat{P} = \begin{pmatrix} p_1 \\ \vdots \\ p_{n_c} \end{pmatrix}$ where $P = \begin{pmatrix} p_1 & \cdots & p_{n_c} \end{pmatrix}$

Then, AP = 0 $\equiv$ $\breve{A}\widehat{P} = 0$

$$A\,(\delta P + P^{(0)}) \perp \mathcal{Z} \text{ and } \delta P \in \mathcal{Z} \quad \Leftrightarrow \quad Z^T \breve{A}(Zu + \widehat{P}^{(0)}) = 0$$

where Z is an ortho-normal basis for squeezed $\mathcal{Z}$

Richardson iteration: $u^{(s)} = u^{(s-1)} - \alpha^{(s)}(Z^T \breve{A}\widehat{P}^{(0)} + Z^T \breve{A} Z u^{(s-1)})$

$$\Rightarrow \qquad \widehat{P}^{(s)} = (I - \alpha^{(s)} Z Z^T \breve{A})\widehat{P}^{(s-1)}$$

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Implicit $ZZ^T\widehat{C}$

Sparsity pattern constraint: remove stray nnzs from *C*

Near null space constraint:

- B = **const** $ZZ^T\widehat{C} \Leftrightarrow C_{i,.} = C_{i,.} + sum(C_{i,.}) / nnz(C_{i,.})$
- General $ZZ^T\widehat{C} \Leftrightarrow C_{i,.}^T = C_{i,.}^T - B_{l+1}[B_{l+1}^T B_{l+1}]^{-1} B_{l+1}^T C_{i,.}^T$

- No need to “squeeze” matrices !!
- Relatively inexpensive

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Costs

$$\widehat{P}^{(s)} = (I - \alpha^{(s)} Z Z^T \breve{A}) \widehat{P}^{(s-1)}$$

$$\widehat{R}^{(s)} = \widehat{R}^{(s-1)} (I - \alpha^{(s)} \breve{A} Z Z^T)$$

Costs depend on

– # of iterations, # of near null space constraints, sparsity pattern

1 iteration
- Matrix-matrix multiply: A P
- Remove nnzs beyond desired sparsity pattern
- Force zero rowsums or apply $I - X^T(X X^T)^{-1} X$

block diagonal

- Costs can be mitigated to give practical algorithms
  - Few iterations needed, e.g. Smoothed Aggregation
  - Amortization opportunities: init. guess, reuse of $X X^T$

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MueMat/ MueLu

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sparsity pattern

constraints

→

Solve Galerkin System via a couple of Richardson iterations

→

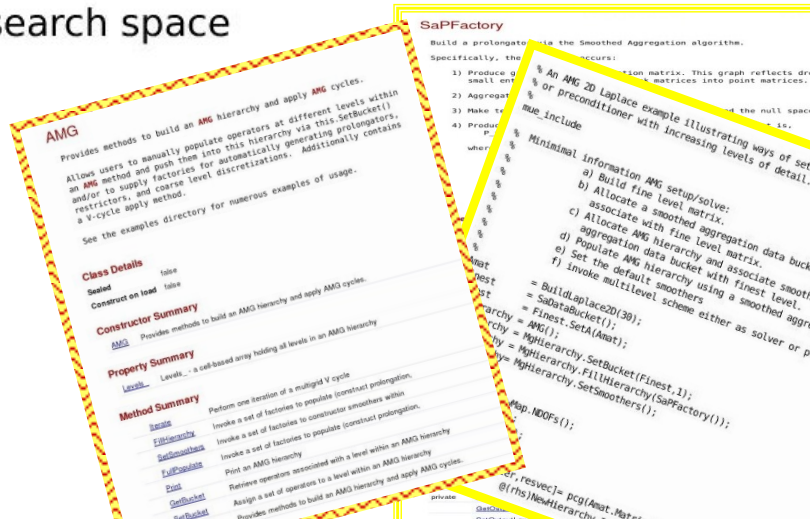
P_k's coefficients

Advantages:

- Flexibility (input):
  - arbitrary coarsening
  - accept any sparsity pattern (arbitrary basis function support)
  - enforce constraints: important modes requiring accurate interpolation
  - choice of norm for minimization and search space
- Robustness

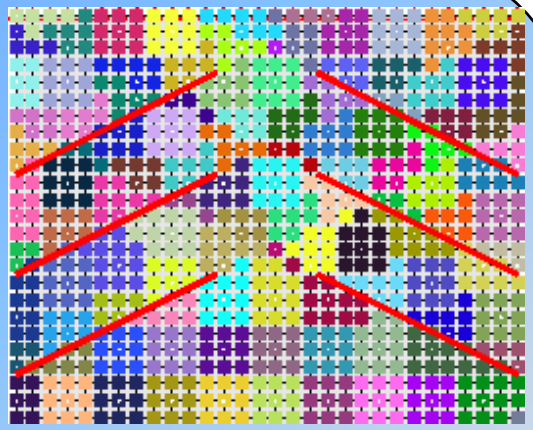
MueLu

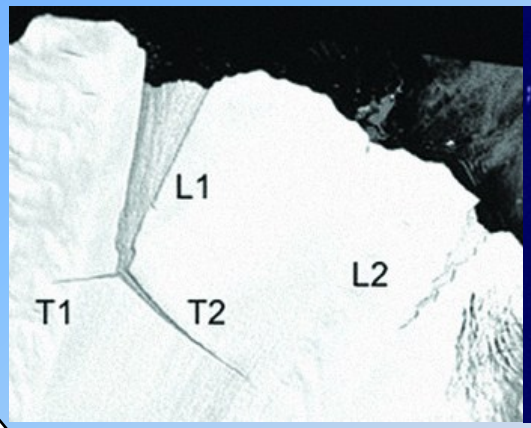
- New algorithms
- Variable block friendly
- Tpetra/Kokkos
  ⇒multicore aware, templated types

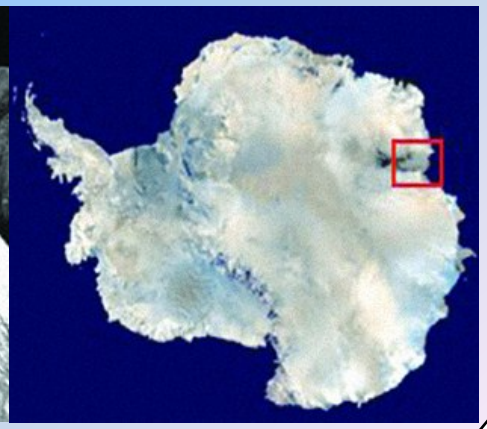


Flexibility/Irregular Sparsity Patterns

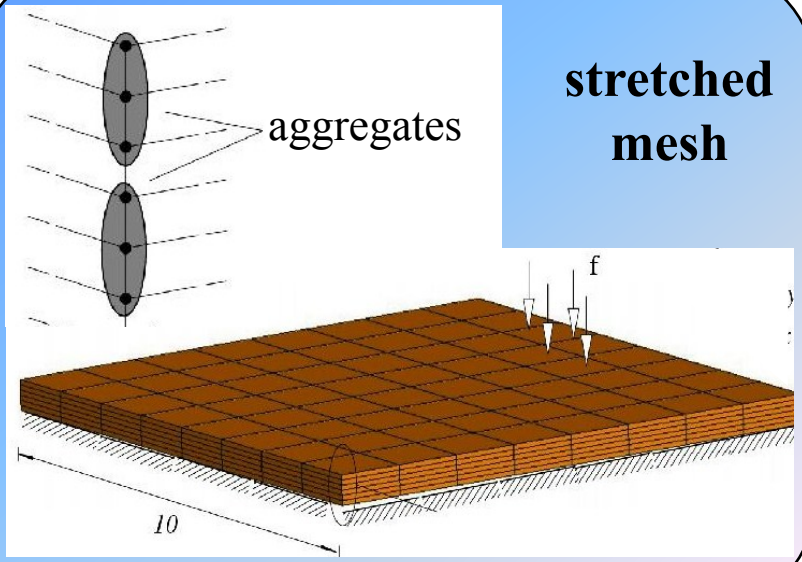
ice fracture via XFEM



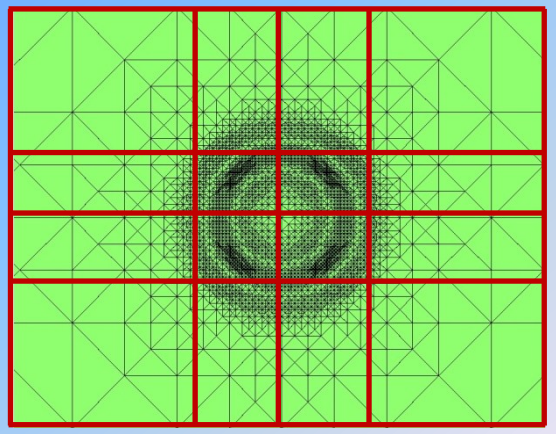




stretched mesh



unstructured ⇒ structured



Results

ice fracture via XFEM

	mesh	vblk	hybrid	Galerkin$^+$
1 crack	30×30	24	22	11
	60×60	24	29	12
	90×90	40	37	18
	120×120	38	42	17
6 cracks	30×30	89	81	16
	60×60	103	116	15
	90×90	134	143	17
	120×120	151	165	16

complexity: $\dfrac{\sum_i nnz(A_i)}{nnz(A)}$

stretched mesh

Mesh	$\epsilon = 1$				$\epsilon = 100$			
	SA		Emin		SA		Emin	
10^3	6	1.30	7	1.07	9	3.21	8	1.24
15^3	8	1.19	9	1.05	12	2.54	12	1.16
20^3	8	1.24	9	1.06	13	3.05	10	1.20
25^3	9	1.26	8	1.07	14	3.04	10	1.20
30^3	10	1.22	11	1.05	15	3.06	13	1.19
35^3	10	1.24	10	1.06	16	3.03	13	1.19
40^3	10	1.26	9	1.06	16	3.21	11	1.21

unstructured $\Rightarrow$ structured

domains	Galerkin$^+$
4×4	12
8×8	11
12×12	10
16×16	10

$A \neq A^T$ AMG results

Transfer Op	κ				complexity
	400^{-1}	800^{-1}	1600^{-1}	3200^{-1}	
PA-AMG	44	58	80	106	1.143
PG-AMG	26	40	58	80	1.217
Galerkin(1,0.95)	30	43	59	80	1.181
Galerkin(1,1.05)	30	41	58	78	1.186
Galerkin(2,0.95)	28	39	53	72	1.181
Galerkin(2,1.05)	27	37	52	71	1.186
Galerkin(3,0.95)	26	36	50	66	1.181
Galerkin(3,1.05)	25	35	48	65	1.186

$$-\kappa \Delta u + a(x,y) \cdot \nabla u = 0$$

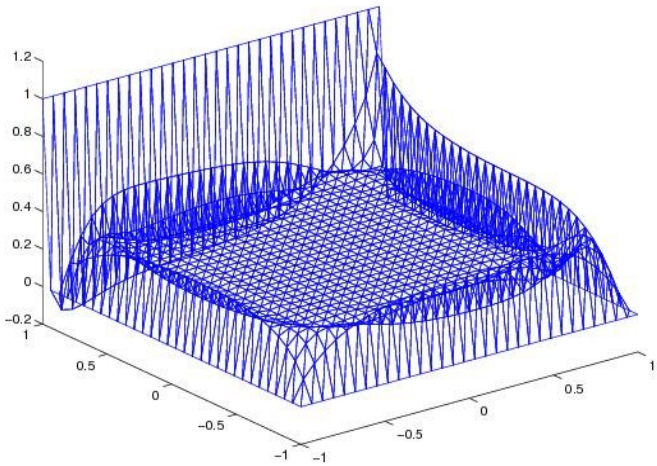
$$a(x,y) = \begin{pmatrix} 2y(1-x^2) \\ -2x(1-y^2) \end{pmatrix}$$

$$u(x,1) = 1, \quad u(x,-1) = 0,$$
$$u(\pm 1, y \neq 0) = 0$$

GMRES its for 10^{-7} residual reduction

AMG Vcycles (3 GS, 3 GS)

64×64 mesh



Conclusions

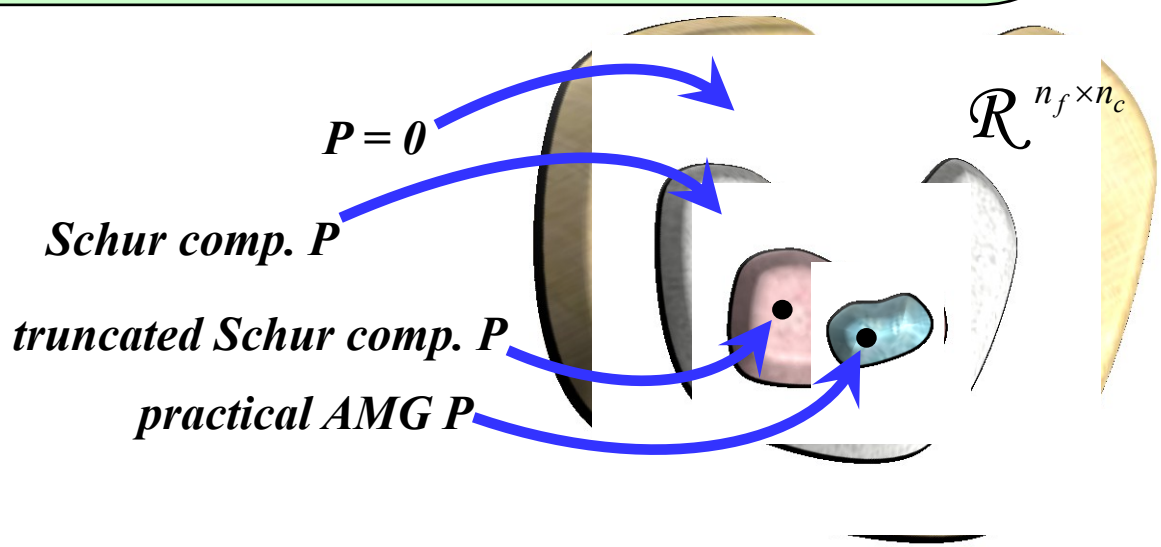
Grid Transfers via Galerkin Projection & Iteration

$$\widehat{P}^{(s)} = (I - \alpha^{(s)} Z Z^T \breve{A}) \widehat{P}^{(s-1)}$$

$\Rightarrow$ Symmetric/Nonsymmetric

$\Rightarrow$ Broad algorithmic framework encompassed by software base

- Reusability
- Flexibility
  - coarsening
  - grid transfer sparsity patterns
  - mode preservation

demonstrations

- XFEM fracture
- anisotropic elasticity
- unstructured $\Rightarrow$ structured
- fluid flow