Agglomeration-based H(curl) and H(div) coarse spaces with approximation properties

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Motivation

We are interested in solving variety of PDEs
(Laplace, elasticity, Darcy, Maxwell, Brinkman - combination of
Darcy and Stokes, ...).
Our goal is two-fold:
We want to discretize the PDE on a general unstructured fine
mesh that resolves the PDE coefficients, and then to come up with
hierarchy of discretization problems that can be used, both

- to build scalable multigrid solver, and/or
- as coarse discretization (upscaling) tool

Motivation, continued

To achieve this goal we develop specialized "*element-based*" algebraic multigrid (AMGe).

We extend previous result:

Pasciak, J., Vassilevski, P. "Exact de Rham Sequences of Spaces Defined on Macro-elements in Two and Three Spatial Dimensions", 2007

such that now, our coarse space have "*guaranteed*" "*approximation properties*".

This is ensured by incorporating enough functions (e.g., vector constants) into the coarse spaces.

In this presentation, we consider applications to Darcy equation and outline an approach for Maxwell equations.

Problem setup

- 3-dimensional polyhedral domain Ω

- Unstructured tetrahedral mesh (extension to hexahedral elements seems possible)

- Sequence of spaces: lowest order Nedelec $\xrightarrow{\nabla\times}$ lowest order Raviart-Thomas $\xrightarrow{\nabla\cdot}$ discontinuous piecewise-constant functions

- Notation (same sequence re-written using symbols):
  $\widetilde{Q}_h \xrightarrow{\nabla\times} \widetilde{R}_h \xrightarrow{\nabla\cdot} \widetilde{M}_h \to 0$

- If the domain Ω is homeomorphic to a sphere, then the sequence above is exact, that is $\text{Null}(\nabla\cdot) = \text{Range}(\nabla\times)$ and $\widetilde{M}_h = \text{Range}(\nabla\cdot)$

- On each tetrahedral element the Nedelec and Raviart-Thomas spaces locally interpolate (contain) vector constants

Problem setup, continued

- The Nedelec and Raviart-Thomas spaces approximate the smooth vector functions with the error linear in mesh parameter h (in H(curl) and H(div) norms, respectively)

- We seek to construct the "coarse" subspaces $\widetilde{Q}_H \subset \widetilde{Q}_h$, $\widetilde{R}_H \subset \widetilde{R}_h$, and $\widetilde{M}_H \subset \widetilde{M}_h$, such that:
    - The sequence $\widetilde{Q}_H \xrightarrow{\nabla \times} \widetilde{R}_H \xrightarrow{\nabla \cdot} \widetilde{M}_H \to 0$ is exact
    - The vector constants (or other suitable functions) are still interpolated "locally" in some sense

- We expect the resulting coarse subspaces to inherit, at least partially, the good approximation properties of the fine-grid spaces

# Element agglomeration and coarse topology

- We group fine elements into non-overlapping agglomerates (we use METIS for partitioning, other approaches are possible)

- Agglomerates play the role of coarse elements

- We build coarse faces as unions of fine faces (triangles). The fine faces which are incident to the same two agglomerates form a coarse face.

- We build coarse edges as unions of fine edges (segments). The fine edges which are incident to the same set of coarse faces (the set must contain two or more coarse faces) form the coarse edge.

Requirements on coarse topology

- ► We require the coarse elements and coarse faces to be "connected" in certain sense
- ► This can be easily ensured by splitting each coarse element and then each coarse face into connected components
- ► In our tests, the splitting did not increase the number of coarse elements or faces substantially (increased by at most 10%)

Coarse elements—illustration
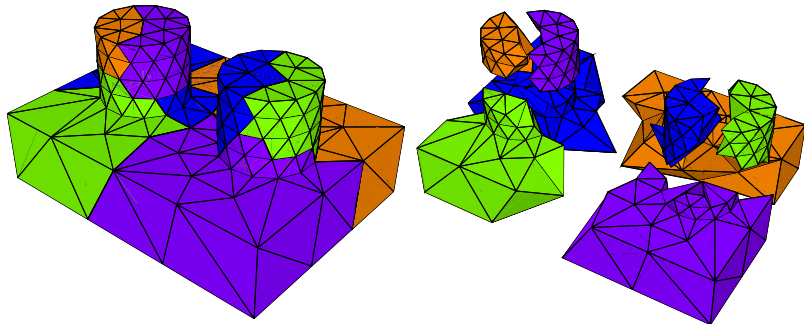
$\widetilde{M}_H$ is defined to consist of functions which are constant on each agglomerate

Constructing the coarse Raviart-Thomas space, $\widetilde{R}_H$

- For each coarse face F consider the restrictions of 3 vector constants $\mathbf{e}_i$ to F
- $\mathbf{e}_1 = (1,0,0)^T$, $\mathbf{e}_2 = (0,1,0)^T$, $\mathbf{e}_3 = (0,0,1)^T$
- "Restriction" means we evaluate fine degrees of freedom associated with F on each vector constant and form the $|F| \times 3$ matrix
- We eliminate linearly dependent columns (e.g., if the coarse face is flat, there will be only one linearly independent column)
- Each remaining column defines a coarse shape function

Constructing the coarse Raviart-Thomas space, $\widetilde{R}_H$, continued

- Coarse shape functions are obtained by solving local mixed system on each agglomerate:

$$\begin{aligned}
(\mathbf{r}, \phi)_T + (p, \nabla \cdot \phi)_T \quad &= 0, \quad \forall \phi \in R_h(T) \\
(\nabla \cdot \mathbf{r}, \theta)_T \quad &= 0, \quad \forall \theta \in M_h(T) \\
\mathbf{r} \cdot \mathbf{n} \text{ is given on } \partial T.
\end{aligned}$$

- $R_h(T)$ is the restriction of $\widetilde{R}_h$ to the agglomerate T with the additional constraint that for each $\mathbf{r} \in R_h(T)$ $\mathbf{r} \cdot \mathbf{n} = 0$ on ∂T

- $M_h(T)$ is the restriction of $\widetilde{M}_h$ to the agglomerate T with the additional constraint that each $q \in M_h(T)$ has zero average on T

- In other words, we are locally solving the mixed formulation of the Laplacian with constant source function and pure-flux boundary conditions

Properties of coarse Raviart-Thomas space $\widetilde{R}_H$

- Divergence of $\widetilde{R}_H$ coincides with $\widetilde{M}_H$ ($\widetilde{M}_H$ is the space of functions which are constant on each agglomerate)
  - Technically, an assumption is required that each coarse face has non-zero "average normal"
  - This condition can be easily and cheaply ensured (e.g. by splitting the offending coarse faces)
- Vector constants are interpolated exactly on each coarse element
- If coarse elements are tetrahedrons, we exactly recover the standard Raviart-Thomas space

A variation of $\widetilde{R}_H$ construction for flows in porous media

- Consider mixed formulation of Darcy flow in Ω, using $\widetilde{R}_h$ and $\widetilde{M}_h$ as discretization spaces: find $u \in \widetilde{R}_h$ and $p \in \widetilde{M}_h$ such that

$$(K\mathbf{u}, \mathbf{v})_\Omega + (p, \nabla \cdot \mathbf{v})_\Omega = 0, \quad \forall \mathbf{v} \in \widetilde{R}_h$$
$$(\nabla \cdot \mathbf{u}, \theta)_\Omega = (f, \theta)_\Omega, \quad \forall \theta \in \widetilde{M}_h$$

- K may have jumps of several orders of magnitude
- The fluxes that one wants to approximate may not locally resemble a vector constant

A variation of $\widetilde{R}_H$ construction for flows in porous media, continued

- ▶ Incorporate the inverse permeability field K into local mixed systems:

$$\begin{aligned}
(K\mathbf{r}, \phi)_T + (p, \nabla \cdot \phi)_T &= 0, \quad \forall \phi \in R_h(T) \\
(\nabla \cdot \mathbf{r}, \theta)_T &= 0, \quad \forall \theta \in M_h(T) \\
\mathbf{r} \cdot \mathbf{n} \text{ is given on } \partial T.
\end{aligned}$$

- ▶ To define coarse shape functions on coarse faces, solve three Darcy flow problems $K\mathbf{u} = \nabla p$, $\nabla \cdot \mathbf{u} = 0$ in some neighborhood O_F of each coarse face F with 3 different pressure boundary conditions: $p|_{\partial O_F} = x, y, z$

- ▶ Restrict the flux parts of the three solutions to F, then proceed as before

- ▶ Note that if $K = I$ then the flux parts of the three solutions will be the three vector constants $\mathbf{e}_i$

Construction of the coarse Nedelec space, $\widetilde{Q}_H$

- We associate up to three degrees of freedom with each coarse edge (by restricting 3 vector constants)
- We additionally associate up to three degrees with each coarse face (as described on following slides)
- Two extension steps: "edge-to-face" and "face-to-interior"

Construction of the coarse Nedelec space: edge-to-face extension

For each coarse face F...

- ▶ We decompose the space of traces of coarse Raviart-Thomas shape functions into the average normal $\mathbf{n}_F$ and remaining orthogonal directions $\mathbf{s}_i$, $i = 1, 2$.
- ▶ Each $\mathbf{s}_i$ has the property that $\int_F \mathbf{s}_i \cdot \mathbf{n} \, dA = 0$
- ▶ Each $\mathbf{s}_i$ gives rise to a coarse shape function with the trace on F defined as a solution of the following mixed system:

$$\mathbf{q}_F \cdot \tau = 0 \text{ on } \partial F$$
$$\exists p \in \mathcal{M}_h : (Z\mathbf{q}_F, \phi) + (\nabla^\perp \cdot \phi, p) = 0 \,\, \forall \phi \in K_h(F)$$
$$\nabla^\perp \cdot q_F = s_i$$

Construction of the coarse Nedelec space, continued

For each coarse face F...

- ► Each coarse shape function associated with coarse edge is extended to each adjacent coarse face by solving the following mixed system:

$$\tilde{\mathbf{q}}_{\text{F}} \cdot \tau|_{\partial F} \text{ is given}$$

$$\exists p \in \mathcal{M}_h(F): \ (Z\tilde{\mathbf{q}}_{\text{F}}, \phi) + (\nabla^{\perp} \cdot \phi, p) = 0 \ \forall \phi \in K_h(F)$$

$$\nabla^{\perp} \cdot \tilde{\mathbf{q}}_{\text{F}} = \mathbf{n}_{\text{F}} \frac{\int_{\partial F} \tilde{\mathbf{q}}_{\text{F}} \cdot \tau \, \mathrm{d}L}{\int_F \mathbf{n}_{\text{F}} \cdot \mathbf{n} \, \mathrm{d}A}$$

- ► For non-planar faces with planar boundary, we associate one extra degree of freedom with the coarse face (average normal)

- ► Final step is face-to-interior extension (done similar to the Raviart-Thomas case)

Numerical results for coarse Raviart-Thomas space

- ► Unstructured mesh on box-plus-cylinder domain
- ► Permeability tensor $K = (1 + 10(x^2 + y^2 + z^2))^{-1}I$
- ► Exact solution $p = \sin(\pi x)\sin(\pi y)\sin(\pi z)$
- ► METIS agglomeration, approximately 80 fine elements per agglomerate
- ► Consider fine and coarse mixed systems:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{bmatrix} P_u^T A P_u & P_u^T B^T P_p \\ P_p^T B P_u & 0 \end{bmatrix} \begin{bmatrix} u_c \\ p_c \end{bmatrix} = \begin{bmatrix} 0 \\ P_p^T f \end{bmatrix}$$

- ► We solve coarse mixed system, interpolate solutions (flux and pressure) to the fine grid and measure errors as compared to the exact solutions
- ► We observe how errors diminish as we refine the original mesh (this increases number of agglomerates)

# Numerical results for coarse Raviart-Thomas space

# of refinements	flux L_2 error	flux $H(\text{div})$ error	pressure L_2 error
0	0.73	0.78	0.81
1	0.46	0.57	0.62
2	0.23	0.33	0.37
3	0.099	0.19	0.21

The finest mesh has 354304 elements and 720192 faces.

Numerical results for SPE10 data set

- ► Rectangular grid, $60 \times 220 \times 85$ cells (1122000 cells total)
- ► Permeabilities along three coordinate axes are given for each cell
- ► Permeabilities range from 10^{-4} to 10^7
- ► We run 2-level method
- ► Overlapping Schwarz is used as a smoother
- ► After first iteration, flux error is divergence-free (i.e. we are effectively solving linear system with SPD matrix)
- ► We coarsen uniformly in x- and y- directions (not in z-direction)
- ► We achieve asymptotic convergence rate (for stationary iteration) around 0.68

SPE10 data set – graph of permeability in Z-direction
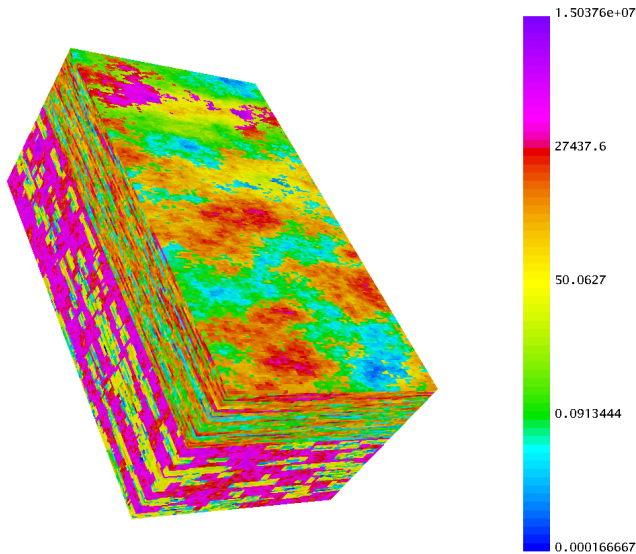
Conclusions and future work

Our spaces have better approximation properties than the earlier work, specifically:

- ▶ We have built coarse versions of Raviart-Thomas spaces (right-most part of the de Rham sequence)
- ▶ We have proposed the construction of coarse Nedelec space (middle of the de Rham sequence)
- ▶ The coarse sequence retains the exactness of the fine one

Work in progress:

- ▶ Complete the de Rham sequence (i.e., build coarse version of H^1-conforming space)
- ▶ Extend to next-to-lowest order space and higher
- ▶ Apply the algorithm recursively for multilevel coarsening