A New Approximate Block Factorization Preconditioner for 2D Incompressible Resistive Magnetohydrodynamics

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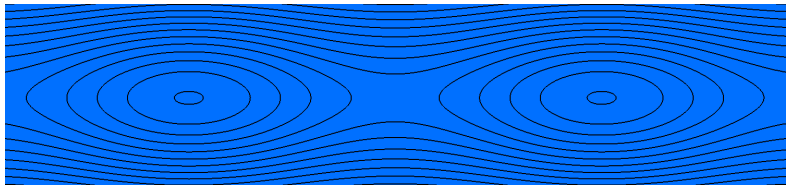
Multiple Time Scales: Magnetohydrodynamics



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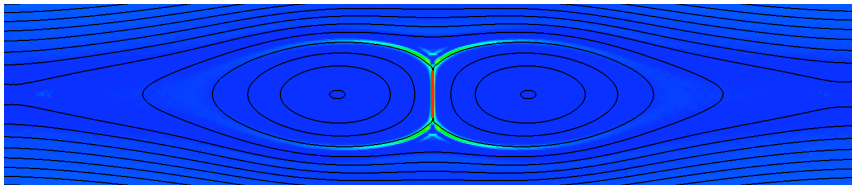

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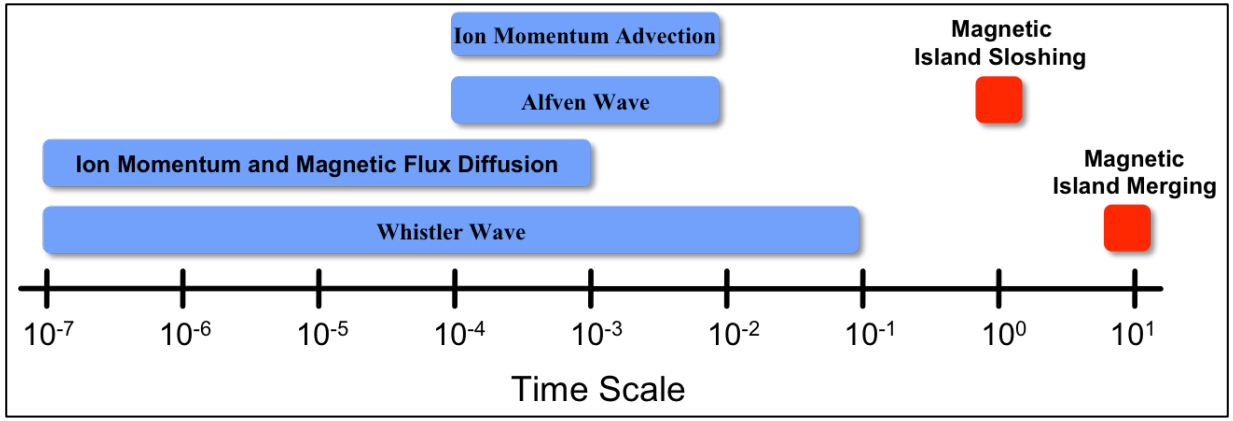
Magnetic island formation/coalescence

➔ Can be modeled by Extended MHD

Timescales for Magnetohydrodynamics



Multiple Time Scales

MHD times scales difficult for explicit, operator-split, and semi-implicit integration

- Fast modes prohibit explicit simulation
  - × Stability restrictions imply small time steps
  - × For long time integration accuracy becomes problematic
- Interacting time scales make semi-implicit and operator split methods challenging

Stable long time scale integration can be enabled by implicit time stepping

- Must solve linear system: Newton's Method (one iteration)

$$\text{Solve } \mathbf{J}p_k = -F(x_k) \text{ where } \mathbf{J} = \partial F/\partial x$$

$$x_{k+1} = x_k + p_k$$

Our approach is to solve using preconditioned Newton-Krylov methods

- Effective preconditioning is key to parallel scalability

What must a preconditioner do?

- Preconditioner is key to computational efficiency and parallel scalability
- What must a preconditioner do?
  1. Handle ill conditioning of system due to:

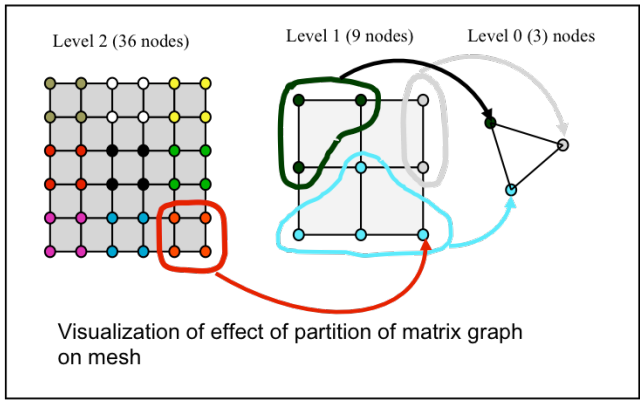
     Fast waves, advection, elliptic operators, …

     Multiphysics systems strongly couple mechanisms, producing multiple time- and length-scales
  2. Must optimally scale with increasing:
     - problem size
     - processor count
- For incompressible MHD specifically
  1. Pressure-Velocity coupling: incompressibility constraint
  2. Alfven Wave: Velocity-Magnetics coupling
  3. Material advection (flow velocity)
  4. Dissipative operators (momentum, magnetics)

Preconditioning

Three flavors of preconditioning

1. Domain Decomposition
   - ILU Factorization on each processor (with overlap)
2. Multilevel methods: ML Library (Tuminaro, Sala, Hu, Siefert, Gee)
   - Smoothed Aggregation
   - Aggressive Coarsening (AggC):
     - 3-Level method
     - Multiple unknowns per node
     - Aggregation rate chosen to fix coarse grid size
     - ILU solvers to fix aggressive aggregation
3. Block Preconditioners



Aggregation based Multigrid:
Vanek, Mandel, Brezina, 1996; Vanek, Brezina, Mandel, 2001; Sala, Formaggia, 2001

Block preconditioning: CFD example

Consider discretized Navier-Stokes equations

$$\left.\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\nabla^2\mathbf{u} + \nabla p = f \\ \nabla\cdot\mathbf{u} = 0\end{aligned}\right\} \Leftrightarrow \begin{bmatrix} F & B^T \\ B & C \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

Fully Coupled Jacobian

$$\mathcal{A} = \begin{bmatrix} F & B^T \\ B & C \end{bmatrix}$$

Block Factorization

$$\mathcal{A} = \begin{bmatrix} I & \\ BF^{-1} & I \end{bmatrix}\begin{bmatrix} F & B^T \\ & S \end{bmatrix}$$

$$S = C - BF^{-1}B^T$$

- Coupling in Schur-complement

Preconditioner

$$\mathcal{A}^{-1} \approx \mathcal{M}^{-1} = \begin{bmatrix} \hat{F} & B^T \\ & \hat{S} \end{bmatrix}^{-1}$$

Required operators:

- $F^{-1} \approx \hat{F}^{-1} \rightarrow$ Multigrid
- $S^{-1} \approx \hat{S}^{-1} \rightarrow$ PCD, LSC, SIMPLEC

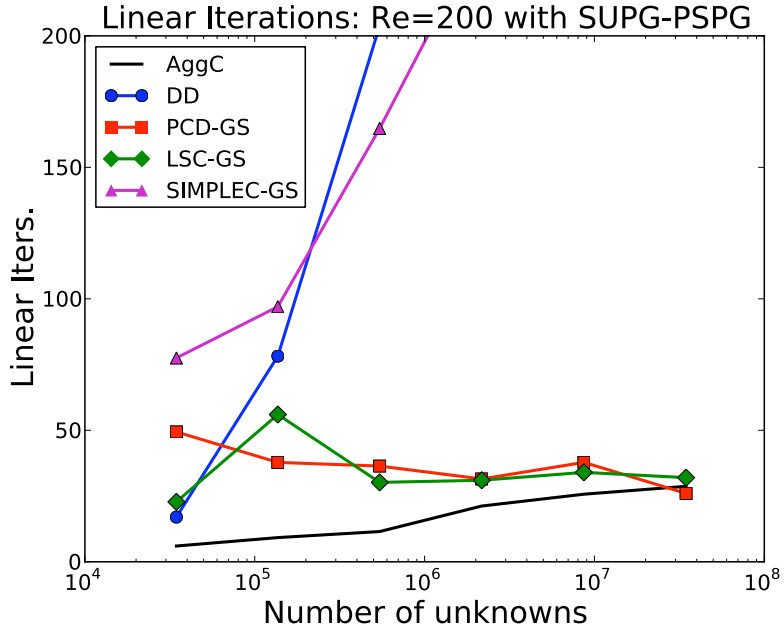
Properties of block factorization

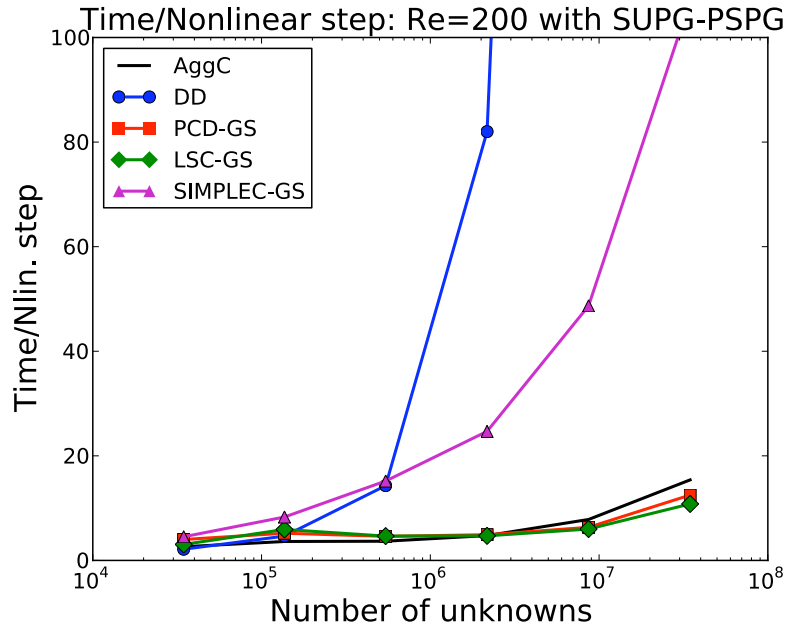
1. Important coupling in Schur-complement
2. Better targets for AMG $\rightarrow$ leveraging scalability

Properties of approximate Schur-complement

1. "Nearly" replicates physical coupling
2. Invertible operators $\rightarrow$ good for AMG

CFD Weak Scaling: Steady Backward Facing Step





Fully coupled Algebraic

AggC: Aggressive Coarsening Multigrid

DD: Additive Schwarz Domain Decomposition

Block Preconditioners

PCD & LSC: Commuting Schur complement

SIMPLEC: "Physics-based" Schur complement

Take home: Block preconditioners competitive with fully coupled multigrid for CFD

* Paper accepted: E. C. Cyr, J. N. Shadid, R. S. Tuminaro, Stabilization and Scalable Block Preconditioning for the Navier-Stokes Equations, Accepted by J. Comp. Phys., 2011.

Incompressible MHD: 2D Vector Potential Formulation

Magnetohydrodynamics (MHD) equations couple fluid flow to Maxwell's equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\nabla^2\mathbf{u} + \nabla p + \nabla\cdot\left(-\frac{1}{\mu_0}\mathbf{B}\otimes\mathbf{B} + \frac{1}{2\mu_0}\|\mathbf{B}\|^2\mathbf{I}\right) = f$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\frac{\partial A_z}{\partial t} + \mathbf{u}\cdot\nabla A_z - \frac{\eta}{\mu_0}\nabla^2 A_z = -E_z^0$$

where $\mathbf{B} = \nabla\times\mathbf{A}$, $\mathbf{A} = (0, 0, A_z)$

Discretized using a stabilized finite element formulation

Incompressible MHD: Discrete Formulation

Stabilized finite element method in residual form

Momentum	$\mathbf{F}_{m,i} = \int_\Omega \Phi \mathbf{R}_{m,i} d\Omega + \sum_e \int_{\Omega_e} \hat{\tau}_m (\mathbf{v} \cdot \nabla \Phi) \mathbf{R}_{m,i} d\Omega$
Total Mass	$F_P = \int_\Omega \Phi R_P d\Omega + \sum_e \int_{\Omega_e} \hat{\tau}_m \nabla \Phi \cdot \mathbf{R}_m d\Omega$
Z-Vector Potential	$F_{A_z} = \int_\Omega \Phi R_{A_z} d\Omega + \sum_e \int_{\Omega_e} \hat{\tau}_{A_z} (\mathbf{v} \cdot \nabla \Phi) R_{A_z} d\Omega$

Structure of discretized Incompressible MHD system is

$$\mathcal{J}\mathbf{x} = \begin{bmatrix} F & B^T & Z \\ B & C & 0 \\ Y & 0 & D \end{bmatrix} \begin{bmatrix} u \\ p \\ A \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ e \end{bmatrix}$$

Matrices F and D are transient convection operators, C is stabilization matrix

Nested Schur Complements

Block LU factorization gives

$$\begin{bmatrix} F & B^T & Z \\ B & C & 0 \\ Y & 0 & D \end{bmatrix} = \begin{bmatrix} I & & \\ BF^{-1} & I & \\ YF^{-1} & -YF^{-1}B^TS^{-1} & I \end{bmatrix} \begin{bmatrix} F & B^T & Z \\ & S & -BF^{-1}Z \\ & & P \end{bmatrix}$$

where

$$S = C - BF^{-1}B^T$$
$$P = D - YF^{-1}(I + B^TS^{-1}BF^{-1})Z$$

- 3x3 system leads to nested Schur complements
- Nesting is independent of ordering (C^{-1} doesn't exist!)
- How is P approximated?
- Chacon & Knoll explored compressible flow and incompressible flow using stream function vorticity

SIMPLE Motivated Preconditioner

$$\mathcal{M} = \begin{bmatrix} F & B^T & Z \\ & S_{Neu} & -BF^{-1}Z \\ & & P_{Neu} \end{bmatrix}$$

where

$$\begin{aligned} F_{Neu} &= \mathbf{AbsRowSum}(F) \\ S_{Neu} &= C - BF_{Neu}^{-1}B^T \\ P_{Neu} &= D - YF_{Neu}^{-1}\,\mathbf{AbsRowSum}(I \\ &\qquad + B^T\mathbf{AbsRowSum}(S_{Neu})^{-1}BF_{Neu}^{-1})Z \end{aligned}$$

Issues

- SIMPLEC Approximation has issues with large CFL
- Not scalable for fixed timesteps

Two Split Preconditioners for MHD

$\hat{x} = \textbf{SplitPrec-NS}(\mathcal{J}, b)$:

$$x^* = \begin{bmatrix} F & & Z \\ & I & \\ Y & & D \end{bmatrix}^{-1} b,$$

$$r^* = b - \mathcal{J}x^*,$$

$$e = \begin{bmatrix} F & B^T & \\ B & C & \\ & & I \end{bmatrix}^{-1} r^*,$$

$$\hat{x} = x^* + e$$

$\hat{x} = \textbf{SplitPrec-MV}(\mathcal{J}, b)$:

$$x^* = \begin{bmatrix} F & B^T & \\ B & C & \\ & & I \end{bmatrix}^{-1} b,$$

$$r^* = b - \mathcal{J}x^*,$$

$$e = \begin{bmatrix} F & & Z \\ & I & \\ Y & & D \end{bmatrix}^{-1} r^*,$$

$$\hat{x} = x^* + e$$

1. Avoids nested Schur complement
2. Split Magnetics-Velocity (MV) from Navier-Stokes (NS)
3. Corresponds to a "split-factorization"

Splitting for MHD

Algorithm corresponds to an Approximate Block Factorization

$$\mathcal{J} \approx \mathcal{M} = \begin{bmatrix} F & & Z \\ & I & \\ Y & & D \end{bmatrix} \begin{bmatrix} F^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} F & B^T & \\ B & C & \\ & & I \end{bmatrix}$$

- Need to compute $\mathcal{M}^{-1}$
- Requires two 2x2 solves
- Navier-Stokes operator well studied
- How to invert Magnetics-Velocity operator

Question: Do we think it will work?

Splitting for MHD

Does splitting make good preconditioner?

$$\mathcal{M} = \begin{bmatrix} F & & Z \\ & I & \\ Y & & D \end{bmatrix} \begin{bmatrix} F^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} F & B^T & \\ B & C & \\ & & I \end{bmatrix}$$

1. Structurally small perturbation

$$\mathcal{M} = \begin{bmatrix} F & B^T & Z \\ B & C & \\ Y & \boxed{YF^{-1}B^T} & D \end{bmatrix}$$

2. Favorable spectrum

$$\mathcal{J}\mathcal{M}^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ K_u & K_p & (I - YF^{-1}B^TS^{-1}BF^{-1}ZP^{-1}) \end{bmatrix}$$

Challenges of splitting: Requires action of two 2x2 inverses

1. Navier-Stokes system – Block preconditioners PCD, LSC, SIMPLEC
2. Magnetics-Velocity system

Approximating Velocity/Magnetics Coupling

$$\begin{bmatrix} F & Z \\ Y & D \end{bmatrix} = \begin{bmatrix} I & \\ YF^{-1} & I \end{bmatrix} \begin{bmatrix} F & Z \\ & P \end{bmatrix}$$

$$\text{where } \; P = D - YF^{-1}Z$$

Strong form commuting condition

$$\nabla A_z \cdot \left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla - \; \nu_F \nabla^2 \right) \approx \left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla - \; \nu_M \nabla^2 \right) \nabla A_z \cdot$$

motivates discrete commuting

$$YQ_u^{-1}F \approx DQ_a^{-1}Y$$

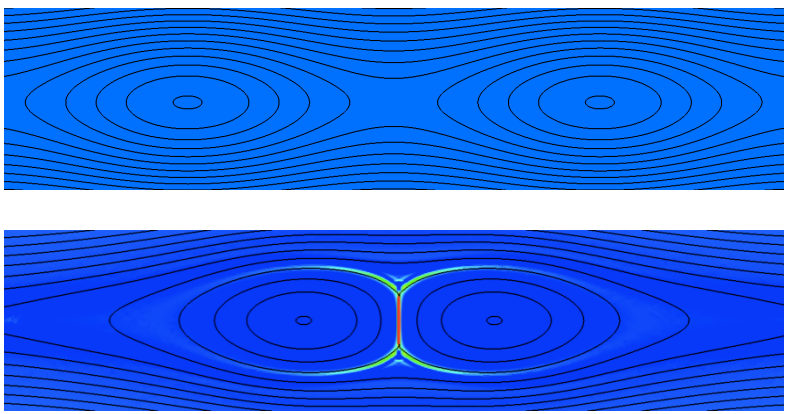
which gives an approximate Schur complement

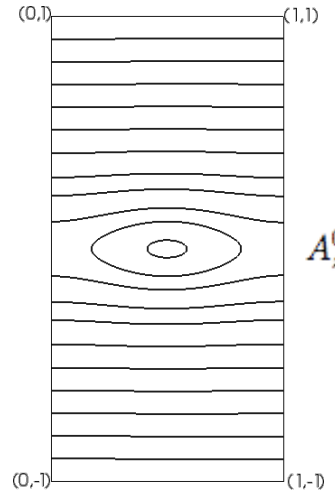
$$D - YF^{-1}Z \approx$$

$$Q_a D^{-1} \left(\underline{DQ_a^{-1}D - YQ_u^{-1}Z} \right)$$

Requires approximate inverse!

Results: Island Coalescence





Simulation on half domain

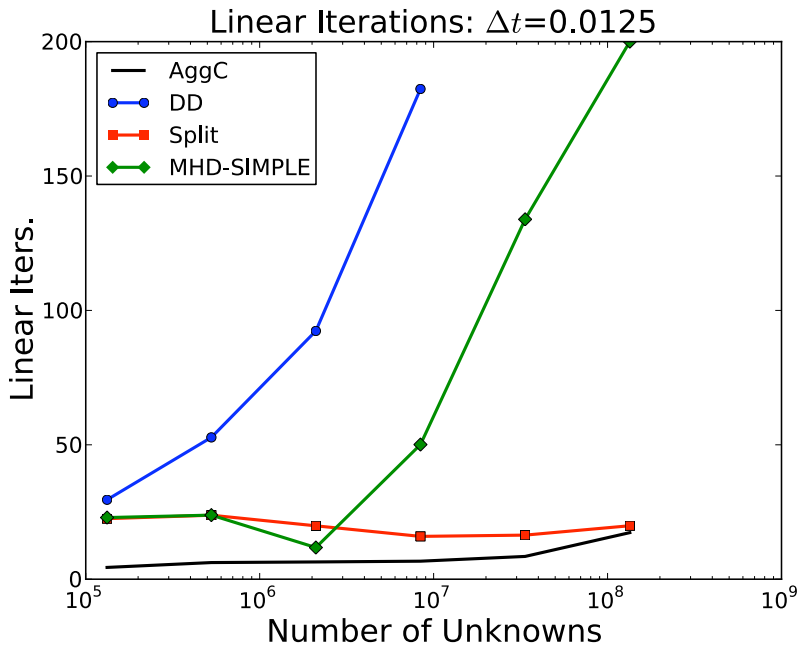
- Symmetry BC
- Perturbed Harris-Sheet

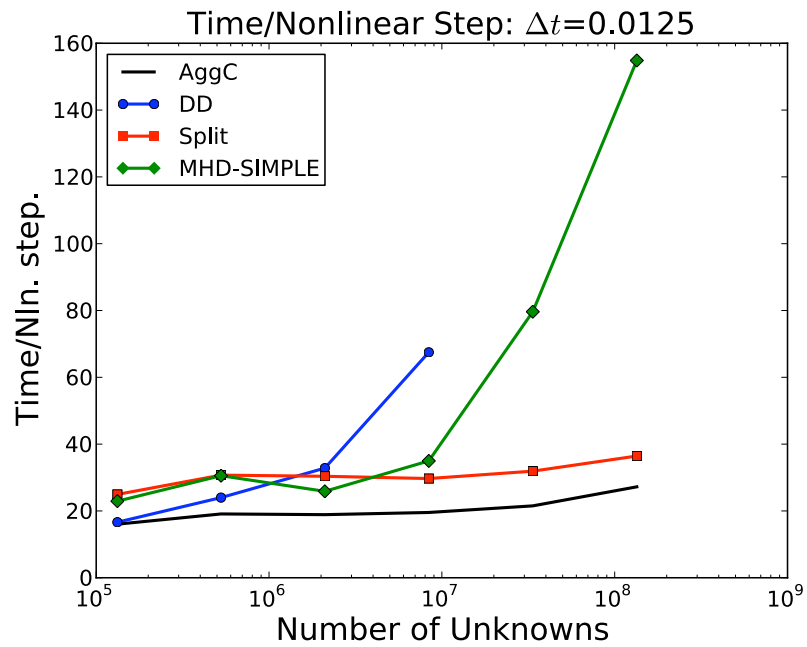
$$A_z^0(x,y,0) = \delta \; ln \left[\cosh\left(\frac{y}{\delta}\right) + \epsilon \; \cos\left(\frac{x}{\delta}\right)\right]$$

Results details

- Lundquist number: 10^4
- Starting time right before reconnection: 5.75s
- Results averaged over 45 uniform timesteps
- Run on 1, 4, 16, 64, 256, and 1024 processors (130000 unks/core)

MHD Weak Scaling: Transient Island Coalescence





Fully coupled Algebraic

AggC: Aggressive Coarsening Multigrid

DD: Additive Schwarz Domain Decomposition

Block Preconditioners

Split: New Operator split preconditioner

SIMPLEC: Extreme diagonal approximations

Take home: Split preconditioner scales algorithmically, more relevant for mixed discretizations

Conclusions

Demonstrated block factorization preconditioners MHD

- Performance not optimized, however results are encouraging
- 3x3 block system has nested Schur complements
- Uses operator splitting approach
  - Separates fluid and magnetics couplings
  - Preconditioner is (structurally) small perturbation of original operator
  - Requires approximating inverse action of two 2x2 operators
  - Weak scaling for fixed timestep indicates scalability with respect to mesh size
  - Possibly good approach for mixed discretizations
- Explored usage of SIMPLEC preconditioner
  - Strong dependence on CFL number
  - Extreme approximations using diagonal approximations