Eigenanalysis for Galerkin Reformulations of Uncertain ODE Systems

Analysis and Reduction of Complex Networks Under Uncertainty

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Spectral Stochastic Representations

Let $\xi : \Omega \to \mathbb{R}^m$ be an L^2 RV on the probability space (Ω, σ, ρ).

Let $\{\varphi_\alpha(\xi) : \alpha = 0, 1, 2, \ldots\}$ be an orthonormal basis of $L^2(\Xi)$.

Let $X : \Xi \to \mathbb{R}$ be $L^2(\Xi)$. It can be represented in the basis φ of $L^2(\Xi)$

$$X(\xi) = \sum_\alpha X_\alpha(a)\, \varphi_\alpha(\xi(\omega))$$

where

$$X_\alpha(a) = \int_\Omega X(a, \omega)\, \varphi_\alpha(\xi(\omega))\, d\rho(\omega) = \langle \varphi_\alpha, X \rangle.$$

Galerkin Reformulation of Uncertain ODEs

Consider a parameterized ODE

$$\dot{x} = f(\xi, x) \qquad x(\xi, 0) = x_0(\xi)$$

with $x(\xi, t) \in \mathbb{R}^n$. Represent x as

$$x(\xi, t) = \sum_{\alpha} x_\alpha(t)\, \varphi_\alpha(\xi)$$

where

$$x_\alpha(t) = \langle \varphi_\alpha(\xi), x(\xi, t) \rangle$$

and so these coefficients are governed by

$$\begin{aligned} \dot{x}_\alpha &= \left\langle \varphi_\alpha(\xi), \frac{d}{dt} x(\xi, t) \right\rangle \\ &= \langle \varphi_\alpha(\xi), f(\xi, x) \rangle \\ \dot{x}_\alpha &= g(x_\beta) \end{aligned}$$

where $\alpha, \beta = 0, \ldots, P$.

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where $\alpha, \beta = 0, \ldots, P$.

Example: Galerkin Reformulation of Uncertain ODEs

Consider the ODE

$$\dot{x} = x(9 - x^2)$$

with random initial condition

$$x(\xi, 0) = a + b\xi.$$

The coefficients of the second-order truncation of the polynomial representation of $x(\xi, t)$ is governed by the IVP

$$\begin{aligned}
\dot{x}_0 &= 9x_0 - x_0^3 - 3x_0x_2^2 - 2\sqrt{2}x_2^3 - 3x_0x_1^2 - 3\sqrt{2}x_1^2x_2, \\
\dot{x}_1 &= 9x_1 - 3x_1^3 - 3x_0^2x_1 - 6\sqrt{2}x_0x_1x_2 - 15x_1x_2^2, \\
\dot{x}_2 &= 9x_2 - 15x_2^3 - 3x_0^2x_2 - 6\sqrt{2}x_0x_2^2 - 3\sqrt{2}x_0x_1^2 - 15x_2x_1^2 \\
& x_0(0) = a, \qquad x_1(0) = b, \qquad x_2(0) = 0
\end{aligned}$$

Jacobian of Uncertain System

The dynamical system can be locally characterized by the eigenstructrure of the Jacobian matrix. The enteries of the Jacobian matrix J of the sampled system is given by

$$J_{ij}(\xi, t) = \frac{\partial f^i}{\partial x^j}(\xi, x(\xi, t))$$

At each fixed time t, $J(\xi, t)$ is a random matrix.

Jacobian Matrix of Reformulated System

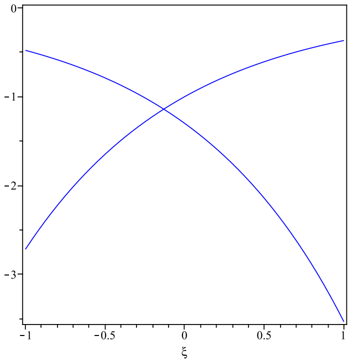
The Jacobian matrix of the coefficient system can be thought of as a block matrix with blocks

$$\begin{aligned}\mathcal{J}_{\alpha\beta}(t) &= D_{x_\beta} \int_\Xi f(\xi, x(\xi, t))\, \varphi_\alpha(\xi)\, d\mu(\xi) \\ &= \int_\Xi \varphi_\alpha(\xi)\, J(\xi, t)\, \varphi_\beta(\xi)\, d\mu(\xi)\end{aligned}$$

Truncate the representation so that $\alpha, \beta = 0, \ldots, P$.
$\mathcal{J}$ is then a $n(P+1) \times n(P+1)$ matrix.

Spectrum of sampled system

For each value of ξ, the Jacobian $J(\xi)$ has a spectrum $S(\xi)$ with n eigenvalues.



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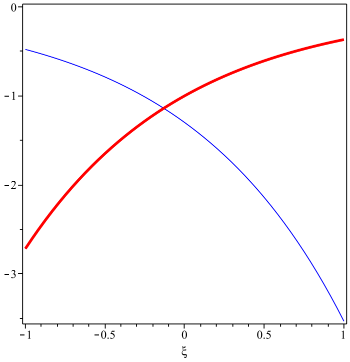
$$\det(J(\xi) - I\lambda(\xi)) \equiv 0.$$



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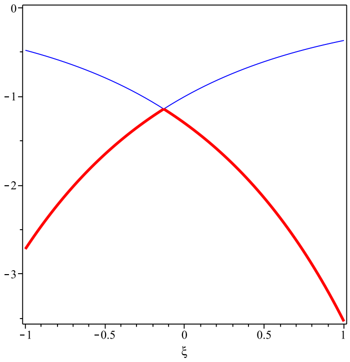

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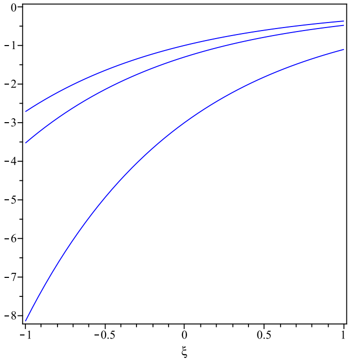

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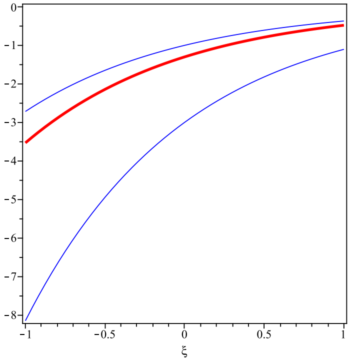

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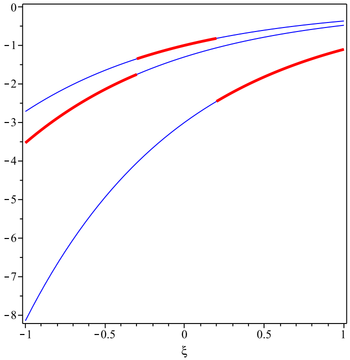


Jacobian

For $n = 1$ the dynamics of the Chaos coefficients are governed by $\dot{x}_k = \langle f(x)\, \varphi_k \rangle$. At a fixed point in time, the Jacobian

$$\mathcal{J}_{\alpha\beta} = \langle \varphi_\alpha,\, f_x\, \varphi_\beta \rangle$$

is symmetric and (real) diagonalizable.

The numerical range of a matrix M is

$$W(M) = \{v^* M v : v \in \mathcal{C}^m, v^* v = \|v\|^2 = 1\}.$$

Note that

$$\mathrm{spect}(M) \subset W(M).$$

Furthermore, let

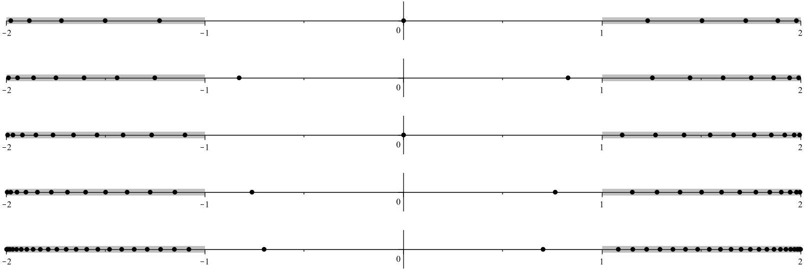
$$\tilde{W}(J) = \bigcup_{\text{a.e. } \xi} W(J(\xi)).$$

Theorem. (Sonday et. al.)

$$\mathrm{spect}\left(\mathcal{J}^P\right) \subset W\left(\mathcal{J}^P\right) \subset \mathrm{conv}\left(\tilde{W}(J)\right)$$

$n = 1, \xi \sim U([-1, 1]), \dot{x} = J(\xi)\, x$

$$J(\xi) = \begin{cases} \xi + 1 & \text{for} \quad \xi \geq 0, \\ \xi - 1 & \text{for} \quad \xi < 0. \end{cases}$$

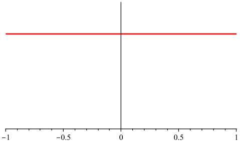


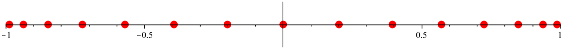
Comparison of Spectra, part 1

Consider the ODE $\dot{x} = \frac{1}{2}x^2, x(\xi, 0) = \xi$. At $t = 0$ the Galerkin Jacobian is

$$\mathcal{J}_{\alpha\beta} = \langle \xi \, \varphi_\alpha(\xi) \, \varphi_\beta(\xi) \rangle$$

and so the spectrum is the locus of the Gauss points.





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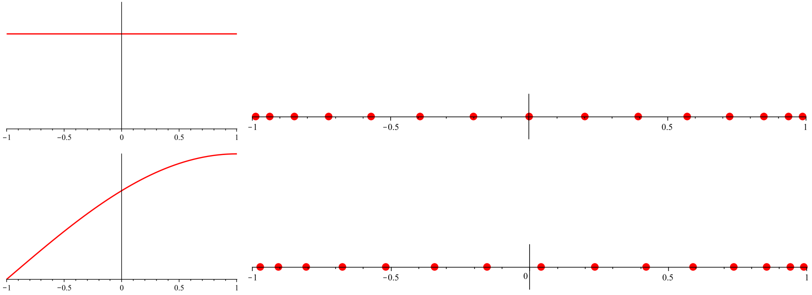
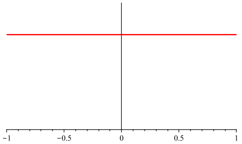


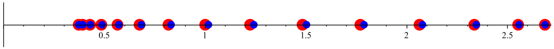
Asymptotic distribution of Gauss points: $\dfrac{1}{\pi} \dfrac{1}{\sqrt{1 - x^2}}$

Comparison of Spectra, part 2

Consider the ODE $\dot{x} = e^x$, $x(\xi, 0) = \xi$. At $t = 0$ the Galerkin Jacobian is

$$\mathcal{J}_{\alpha\beta} = \langle e^{\xi}\, \varphi_\alpha(\xi)\, \varphi_\beta(\xi) \rangle$$





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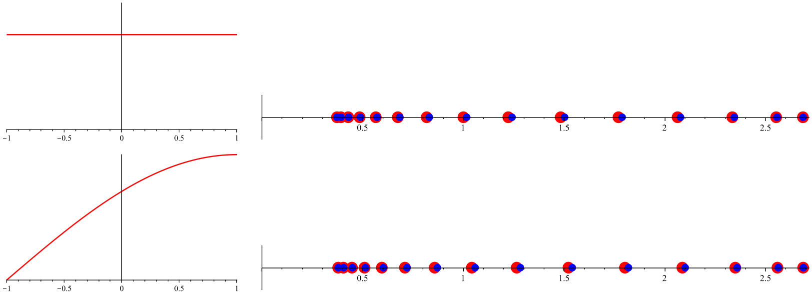

Weak Convergence of Spectra: $n = 1$ case

Theorem (Nevai). Assymptotic Eigenvalue Distribution ($n = 1$)

Let $\{p_\alpha(\xi)\}_{\alpha=0}^{\infty}$ be a polynomial basis orthogonal under $\mu_{[-1,1]}$. Let J be an $L^\infty(\Omega)$ function such that the moments of $J\,d\mu$ are all finite. Let F be a continuous function in an interval containing the essential range of h. Then the eigenvalues $\{\lambda_\alpha\}_{\alpha=0}^{P}$ of the truncated matrix $\mathcal{J}^P$ satisfy

$$\lim_{P\to\infty} \frac{1}{P+1} \sum_{\alpha=0}^{P} F(\lambda_\alpha) = \int_{-1}^{1} \frac{1}{\pi} \frac{F\left(J(x)\right)}{\sqrt{1-x^2}}\, dx.$$

Corollary. Eigenvalue Reconstruction for $n = 1$.

Interpolating Galerkin system eigenvalues approximates random eigenvalues.

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Interpolating Galerkin system eigenvalues approximates random eigenvalues.

Weak convergence of eigenvalues

Recall that the Jacobian matrix of the coefficient system can be thought of as a block matrix with blocks

$$\mathcal{J}_{\alpha\beta}(t) = \langle \varphi_\alpha, J\,\varphi_\beta \rangle.$$

Let $\{\Lambda_{i\alpha}\}$ be the $n(P+1)$ eigenvalues of $\mathcal{J}^P$.

Theorem (Berry et al). Assymptotic Eigenvalue Distribution ($n > 1$)

Let $\mu_{[-1,1]}$ be a (real valued) measure such that $\mu'(x) > 0$ almost everywhere in $[-1, 1]$. Assume the matrix $\mathcal{J}$ is generated by an L^∞ matrix-valued function J. Let G be a holomorphic function in a compact, simply connected $\Omega \subset \mathcal{C}$ containing the essential range of J. Then

$$\lim_{P\to\infty} \frac{1}{n\,(P+1)} \sum_{i,\alpha} G[\Lambda_{i\alpha}] = \frac{1}{n} \sum_i \int_{-1}^{1} \frac{G[\lambda_i(\theta)]\, d\theta}{\pi\sqrt{1-\theta^2}}.$$

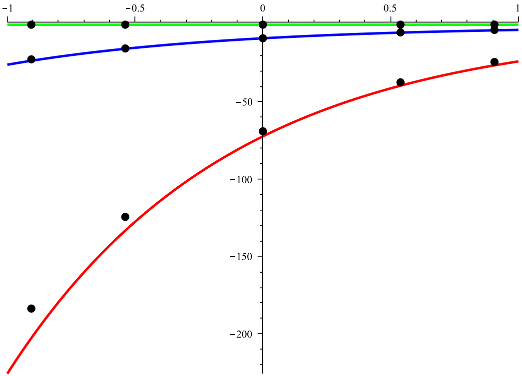
Example: Stiff ODE

Consider the 3-dimensional stiff system (due to Valorani and Goussis)

$$\dot{x} = \frac{5(y^2 - x)}{\varepsilon} + \frac{z - x\,y}{\varepsilon} - x + y\,z$$
$$\dot{y} = -\frac{10(y^2 - x)}{\varepsilon} + \frac{z - x\,y}{\varepsilon} + x - y\,z$$
$$\dot{z} = -\frac{z - x\,y}{\varepsilon} + x - y\,z$$

with initial values $(0.75, 0.75, 0.75)$ and $\varepsilon = 10^{-(\xi+3)/2}$, so $\varepsilon \in [0.01, 0.1]$.

(Example: Stiff ODE, cont.)



Example: ODE with Hopf Bifurcation

The oxidation of CO on a surface can be modeled as (Makeev et al., JCP, 2002)

$$\dot{u} = az - cu - 4duv \qquad \dot{v} = 2bz^2 - 4duv$$

$$\dot{w} = ez - fw \qquad z = 1 - u - v - w$$

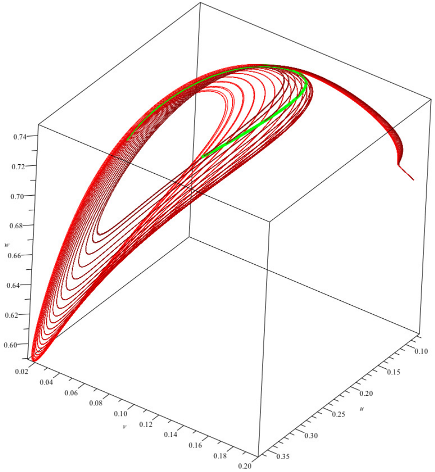
$$a = 1.6, b = 20.75 + .45\xi, c = 0.04, \qquad d = 1.0, e = 0.36, f = 0.016$$

$$u(0) = 0.1, v(0) = 0.2, w(0) = 0.7$$

exhibits Hopf bifurcations for $b \in [20.3, 21.2]$

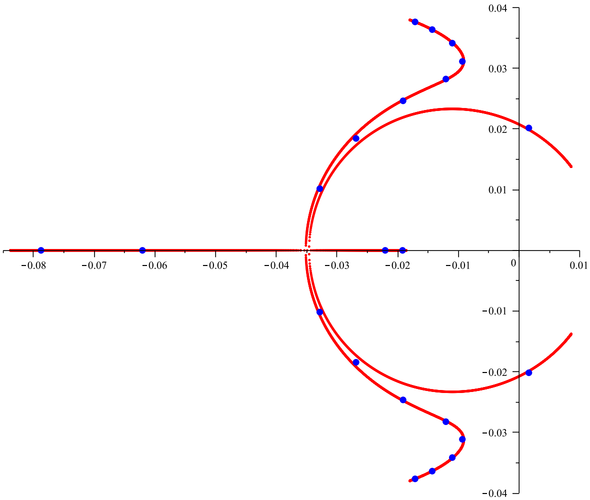
(Example: ODE with Hopf Bifurcation, continued)

Analyzing the stochastic Jacobian at $t = 300$.



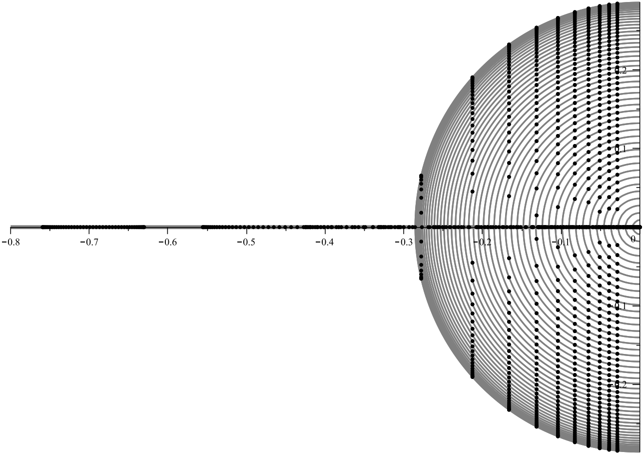
(Example: ODE with Hopf Bifurcation, continued)

PC order 10. Slow eigenvalues.



High-dimensional Example

50 coupled damped linear oscilators. Damping coefficient log-normal.



Eigenvectors

Let $\lambda_{i\alpha}$, $v_{i\alpha}$ be an eigenvalue/vector pair of $\mathcal{J}^P$:

$$\mathcal{J}\, v_{i\alpha} = \lambda_{i\alpha} v_{i\alpha}.$$

Alternatively,

$$\langle \varphi_\beta(\xi), (J(\xi) - \lambda_{i\alpha})\, \nu_{i\alpha}(\xi) \rangle = 0 \qquad \text{for } \beta = 0 \ldots P$$

where $\nu_{i\alpha}(\xi)$ in an n-vector with components

$$\nu_{i\alpha}^k(\xi) = \sum_{\gamma=0}^{P} v_{i\alpha}^{k\gamma}\, \varphi_\gamma(\xi).$$

Eigenpolynomials

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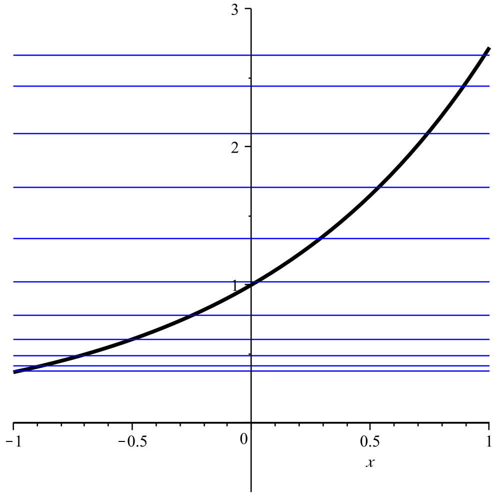
where

$$\nu_\alpha(\xi) = \sum_{\gamma=0}^{P} v_\alpha^\gamma\, \varphi_\gamma(\xi).$$

Local character of PCE Eigenvectors

Polynomial from PCE Eigenvectors "local" to solution of $\lambda(\xi) - \Lambda = 0$.



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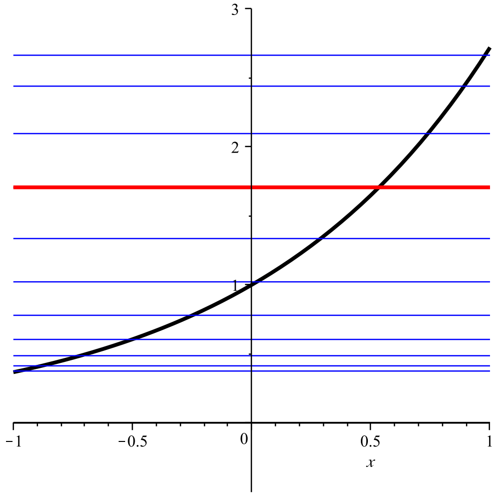

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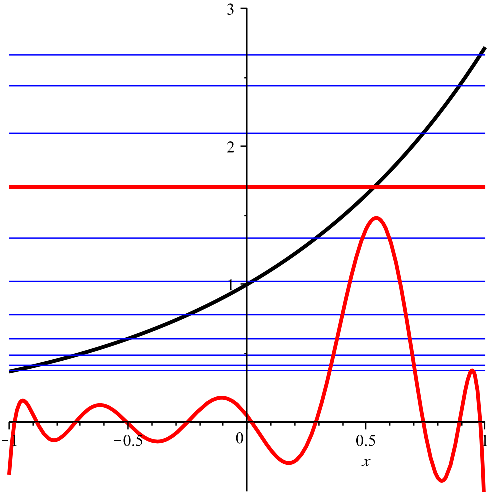

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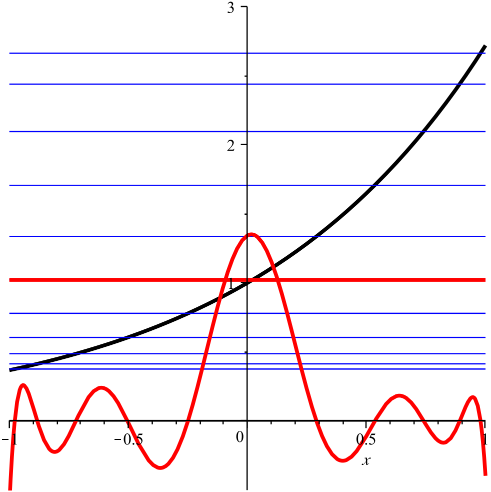

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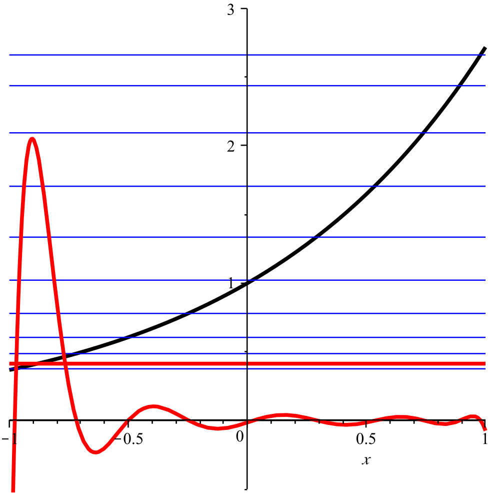
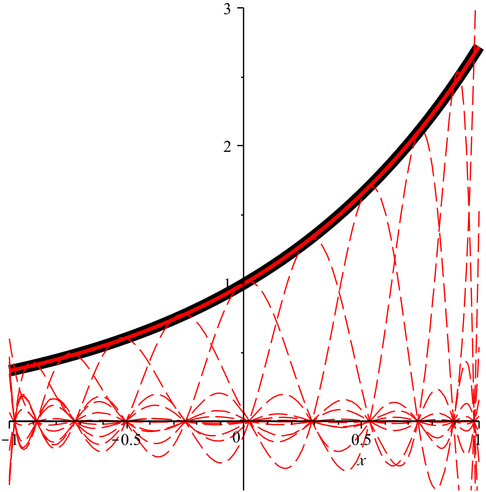
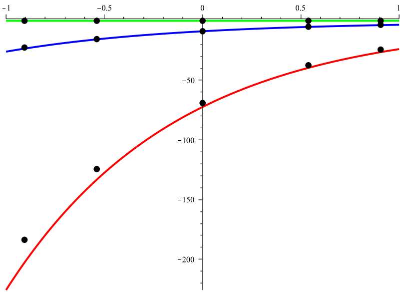


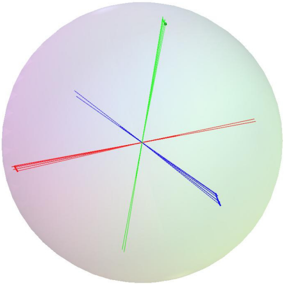
Local character of PCE Eigenvectors

Eigenpolynomials and Eigenvalues can be used to construct the PCE of λ.

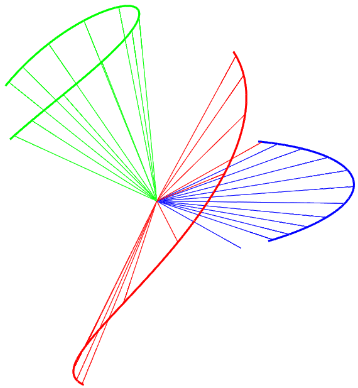


Stiff ODE:
Eigenvector polynomials have local character





Example: Large uncertainties in eigenvectors



Summary

- Comparison of sampled Jacobian and Galerkin Jacobian
- Ambiguities in defining “stochastic eigenvalues”
- When there is a “stochastic eigenvalue” it can be approximated by interpolating the Galerkin system eigenvalues
- Weak convergence of spectra of the two Jacobians
- ξ-local nature of eigenvector polynomials
- When there is a “stochastic eigenvector” it can be approximated by interpolating the expectation of eigenvector polynomials
- The eigenstructure of the Galerkin Jacobian is approximately the union of the eigenstructures of the uncertain system sampled at the Gauss points