Eigenanalysis for Galerkin Reformulations of Uncertain ODE Systems
Analysis and Reduction of Complex Networks Under Uncertainty

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Spectral Stochastic Representations

Let $\xi : \Omega \to \mathbb{R}^m$ be an $L^2$ RV on the probability space $(\Omega, \sigma, \rho)$.

Let $\{\varphi_\alpha(\xi) : \alpha = 0, 1, 2, \ldots\}$ be an orthonormal basis of $L^2(\Xi)$.

Let $X : \Xi \to \mathbb{R}$ be $L^2(\Xi)$. It can be represented in the basis $\varphi$ of $L^2(\Xi)$

$$X(\xi) = \sum_{\alpha} X_\alpha(a) \varphi_\alpha(\xi(\omega))$$

where

$$X_\alpha(a) = \int_{\Omega} X(a, \omega) \varphi_\alpha(\xi(\omega)) \, d\rho(\omega) = \langle \varphi_\alpha, X \rangle.$$
Consider a parameterized ODE

\[ \dot{x} = f(\xi, x) \quad x(\xi, 0) = x_0(\xi) \]

with \( x(\xi, t) \in \mathbb{R}^n \). Represent \( x \) as

\[ x(\xi, t) = \sum_{\alpha} x_\alpha(t) \varphi_\alpha(\xi) \]

where

\[ x_\alpha(t) = \langle \varphi_\alpha(\xi), x(\xi, t) \rangle \]

and so these coefficients are governed by

\[ \dot{x}_\alpha = \langle \varphi_\alpha(\xi), \frac{d}{dt} x(\xi, t) \rangle \]

\[ = \langle \varphi_\alpha(\xi), f(\xi, x) \rangle \]

\[ \dot{x}_\alpha = g(x_\beta) \]

where \( \alpha, \beta = 0, \ldots, P \).
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where \( \alpha, \beta = 0, \ldots, P \).
Consider the ODE

\[ \dot{x} = x(9 - x^2) \]

with random initial condition

\[ x(\xi, 0) = a + b\xi. \]

The coefficients of the second-order truncation of the polynomial representation of \( x(\xi, t) \) is governed by the IVP

\[ \begin{align*}
\dot{x}_0 &= 9x_0 - x_0^3 - 3x_0x_2^2 - 2\sqrt{2}x_2^3 - 3x_0x_1^2 - 3\sqrt{2}x_1^2x_2, \\
\dot{x}_1 &= 9x_1 - 3x_1^3 - 3x_0^2x_1 - 6\sqrt{2}x_0x_1x_2 - 15x_1x_2^2, \\
\dot{x}_2 &= 9x_2 - 15x_2^3 - 3x_0^2x_2 - 6\sqrt{2}x_0x_2^2 - 3\sqrt{2}x_0x_1^2 - 15x_2x_1^2 \\
x_0(0) &= a, \quad x_1(0) = b, \quad x_2(0) = 0
\end{align*} \]
The dynamical system can be locally characterized by the eigenstructure of the Jacobian matrix. The entries of the Jacobian matrix \( J \) of the sampled system is given by

\[
J_{ij}(\xi, t) = \frac{\partial f^i}{\partial x^j}(\xi, x(\xi, t))
\]

At each fixed time \( t \), \( J(\xi, t) \) is a random matrix.
Jacobian Matrix of Reformulated System

The Jacobian matrix of the coefficient system can be thought of as a block matrix with blocks

\[ J_{\alpha \beta}(t) = D_{x_\beta} \int_\Xi f(\xi, x(\xi, t)) \varphi_\alpha(\xi) \, d\mu(\xi) \]

\[ = \int_\Xi \varphi_\alpha(\xi) J(\xi, t) \varphi_\beta(\xi) \, d\mu(\xi) \]

Truncate the representation so that \( \alpha, \beta = 0, \ldots, P \). \( J \) is then a \( n(P + 1) \times n(P + 1) \) matrix.
Spectrum of sampled system

For each value of $\xi$, the Jacobian $J(\xi)$ has a spectrum $S(\xi)$ with $n$ eigenvalues.
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\[
\det(J(\xi) - I\lambda(\xi)) = 0.
\]
For $n = 1$ the dynamics of the Chaos coefficients are governed by
\[ \dot{x}_k = \langle f(x) \varphi_k \rangle. \]
At a fixed point in time, the Jacobian
\[ J_{\alpha\beta} = \langle \varphi_{\alpha}, f_x \varphi_{\beta} \rangle \]
is symmetric and (real) diagonalizable.
The numerical range of a matrix $M$ is

$$W(M) = \{ v^* M v : v \in \mathbb{C}^m, v^* v = \| v \|^2 = 1 \}.$$ 

Note that

$$\text{spect}(M) \subset W(M).$$ 

Furthermore, let

$$\tilde{W}(J) = \bigcup_{(J(\xi))} W(J(\xi)).$$

**Theorem. (Sunday et. al.)**

$$\text{spect}(\mathcal{J}^p) \subset W(\mathcal{J}^p) \subset \text{conv}(\tilde{W}(J))$$
$n = 1, \xi \sim U([-1, 1]), \dot{x} = J(\xi) x$

$$J(\xi) = \begin{cases} 
\xi + 1 & \text{for } \xi \geq 0, \\
\xi - 1 & \text{for } \xi < 0.
\end{cases}$$
Consider the ODE $\dot{x} = \frac{1}{2}x^2$, $x(\xi, 0) = \xi$. At $t = 0$ the Galerkin Jacobian is

$$J_{\alpha\beta} = \langle \xi \varphi_{\alpha}(\xi) \varphi_{\beta}(\xi) \rangle$$

and so the spectrum is the locus of the Gauss points.
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and so the spectrum is the locus of the Gauss points.

Asymptotic distribution of Gauss points:
\[
\frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}}
\]
Consider the ODE \( \dot{x} = e^x \), \( x(\xi, 0) = \xi \). At \( t = 0 \) the Galerkin Jacobian is

\[
J_{\alpha\beta} = \langle e^\xi \varphi_\alpha(\xi) \varphi_\beta(\xi) \rangle
\]
Consider the ODE $\dot{x} = e^x$, $x(\xi, 0) = \xi$. At $t = 0$ the Galerkin Jacobian is

$$J_{\alpha\beta} = \langle e^\xi \varphi_\alpha(\xi) \varphi_\beta(\xi) \rangle$$
Weak Convergence of Spectra: \( n = 1 \) case

**Theorem (Nevai). Assymptotic Eigenvalue Distribution (\( n = 1 \))**

Let \( \{p_\alpha(\xi)\}_{\alpha=0}^\infty \) be a polynomial basis orthogonal under \( \mu_{[-1,1]} \). Let \( J \) be an \( L^\infty(\Omega) \) function such that the moments of \( J \, d\mu \) are all finite. Let \( F \) be a continuous function in an interval containing the essential range of \( h \). Then the eigenvalues \( \{\lambda_\alpha\}_{\alpha=0}^P \) of the truncated matrix \( J^P \) satisfy

\[
\lim_{P \to \infty} \frac{1}{P+1} \sum_{\alpha=0}^P F(\lambda_\alpha) = \int_{-1}^1 \frac{1}{\pi} \frac{F(J(x))}{\sqrt{1-x^2}} \, dx.
\]

**Corollary. Eigenvalue Reconstruction for \( n = 1 \).**

Interpolating Galerkin system eigenvalues approximates random eigenvalues.
Theorem (Nevai). Assymptotic Eigenvalue Distribution \((n = 1)\)

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\]

Corollary. Eigenvalue Reconstruction for \(n = 1\).

Interpolating Galerkin system eigenvalues approximates random eigenvalues.
Recall that the Jacobian matrix of the coefficient system can be thought of as a block matrix with blocks

\[ J_{\alpha \beta}(t) = \langle \varphi_{\alpha}, J \varphi_{\beta} \rangle. \]

Let \( \{\Lambda_{i\alpha}\} \) be the \( n(P + 1) \) eigenvalues of \( J^P \).

**Theorem (Berry et al). Assymptotic Eigenvalue Distribution \((n > 1)\)**

Let \( \mu_{[-1,1]} \) be a (real valued) measure such that \( \mu'(x) > 0 \) almost everywhere in \([-1, 1]\). Assume the matrix \( J \) is generated by an \( L^\infty \) matrix-valued function \( J \). Let \( G \) be a holomorphic function in a compact, simply connected \( \Omega \subset \mathbb{C} \) containing the essential range of \( J \). Then

\[ \lim_{P \to \infty} \frac{1}{n(P+1)} \sum_{i, \alpha} G[\Lambda_{i\alpha}] = \frac{1}{n} \sum_{i} \int_{-1}^{1} \frac{G[\lambda_i(\theta)]}{\pi \sqrt{1 - \theta^2}} d\theta. \]
Consider the 3-dimensional stiff system (due to Valorani and Goussis)

\[
\begin{align*}
\dot{x} &= \frac{5(y^2 - x)}{\varepsilon} + \frac{z - xy}{\varepsilon} - x + yz \\
\dot{y} &= -\frac{10(y^2 - x)}{\varepsilon} + \frac{z - xy}{\varepsilon} + x - yz \\
\dot{z} &= -\frac{z - xy}{\varepsilon} + x - yz
\end{align*}
\]

with initial values \((0.75, 0.75, 0.75)\) and \(\varepsilon = 10^{-\frac{(\xi+3)}{2}}\), so \(\varepsilon \in [0.01, 0.1]\).
The oxidation of CO on a surface can be modeled as (Makeev et al., JCP, 2002)

\[ \dot{u} = az - cu - 4duv \]
\[ \dot{v} = 2bz^2 - 4duv \]
\[ \dot{w} = ez - fw \]
\[ z = 1 - u - v - w \]

\[ a = 1.6, \ b = 20.75 + .45\xi, \ c = 0.04, \ d = 1.0, \ e = 0.36, \ f = 0.016 \]

\[ u(0) = 0.1, \ v(0) = 0.2, \ w(0) = 0.7 \]

exhibits Hopf bifurcations for \( b \in [20.3, 21.2] \)
Analyzing the stochastic Jacobian at $t = 300$. 
PC order 10. Slow eigenvalues.
High-dimensional Example

50 coupled damped linear oscillators. Damping coefficient log-normal.
Eigenvectors

Let $\lambda_{i\alpha}, \nu_{i\alpha}$ be an eigenvalue/vector pair of $\mathcal{J}^P$:

$$\mathcal{J} \nu_{i\alpha} = \lambda_{i\alpha} \nu_{i\alpha}.\$$

Alternatively,

$$\langle \varphi_\beta(\xi), (J(\xi) - \lambda_{i\alpha}) \nu_{i\alpha}(\xi) \rangle = 0 \quad \text{for } \beta = 0 \ldots P$$

where $\nu_{i\alpha}(\xi)$ is an $n$-vector with components

$$\nu^k_{i\alpha}(\xi) = \sum_{\gamma=0}^{P} \nu^{k\gamma}_{i\alpha} \varphi_{\gamma}(\xi).$$
Eigenpolynomials

Let $\lambda_\alpha, \nu_\alpha$ be an eigenvalue/vector pair of $J^P$:

$$J \nu_\alpha = \lambda_\alpha \nu_\alpha.$$ 

Alternatively,

$$\langle \varphi_\beta(\xi), (J(\xi) - \lambda_\alpha) \nu_\alpha(\xi) \rangle = 0 \quad \text{for } \beta = 0 \ldots P$$

where

$$\nu_\alpha(\xi) = \sum_{\gamma=0}^{P} V^\gamma_\alpha \varphi_\gamma(\xi).$$
Local character of PCE Eigenvectors

Polynomial from PCE Eigenvectors “local” to solution of $\lambda(\xi) - \Lambda = 0$. 

![Graph showing the relation between $\lambda(\xi)$ and $\Lambda$.]
Local character of PCE Eigenvectors

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![Graph showing the polynomial and the equation $\lambda(\xi) - \Lambda = 0$.]
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![Graph showing polynomial behavior in the context of PCE Eigenvectors.]
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Local character of PCE Eigenvectors

Eigenpolynomials and Eigenvalues can be used to construct the PCE of $\lambda$. 
Stiff ODE:
Eigenvector polynomials have local character
Example: Large uncertainties in eigenvectors
Summary

- Comparison of sampled Jacobian and Galerkin Jacobian
- Ambiguities in defining “stochastic eigenvalues”
- When there is a “stochastic eigenvalue” it can be approximated by interpolating the Galerkin system eigenvalues
- Weak convergence of spectra of the two Jacobians
- $\xi$-local nature of eigenvector polynomials
- When there is a “stochastic eigenvector” it can be approximated by interpolating the expectation of eigenvector polynomials
- The eigenstructure of the Galerkin Jacobian is approximately the union of the eigenstructures of the uncertain system sampled at the Gauss points