A Scalable Numerical Approach to the Solution of MHD Systems

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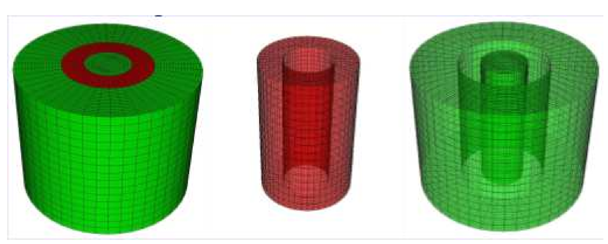
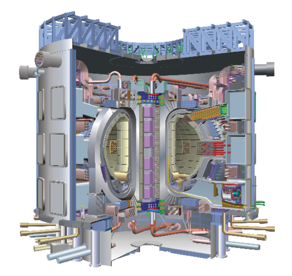
1. A Model MHD (incompressible resistive equations)

MHD models dynamics of electrically conducting fluids (plasmas, liquid metals, electrolytes) and combines fluid dynamics (Navier-Stokes equations) and electromagnetism (Maxwell's equations).

$$\begin{aligned} u_t + (u\cdot\nabla)u - \mu\Delta u + \nabla p - (\nabla\times B)\times B &= 0,\\ \nabla\cdot u &= 0,\\ \partial B_t - \nabla\times(u\times B) + \eta\nabla\times(\nabla\times B) &= 0,\\ \nabla\cdot B &= 0.\end{aligned}$$



3D MHD model problem



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Here η is the resistivity in the system: the Ideal MHD ($\eta = 0$) and Resistive MHD ($\eta > 0$).

2. Some difficulties and solutions

Difficulties:

- Nonlinear convection terms and large coupled systems

Proposed approaches:

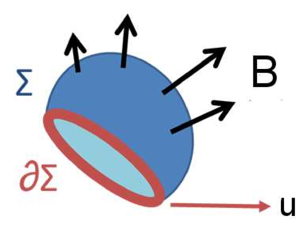
- Describe the convection from geometrical prospective and use Eulerian-Lagrangian Method
- Fast scalable algorithms for the discrete systems

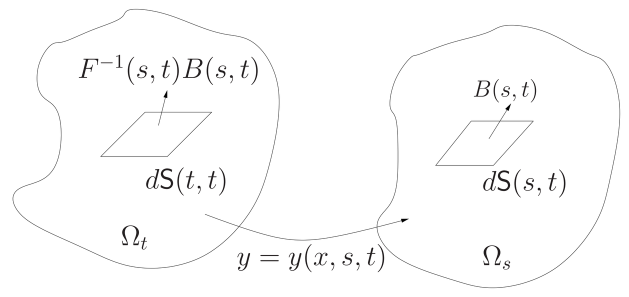
3. Convection in MHD System

For Navier-Stokes equations, nonlinear convection terms come from the derivatives of velocity u (vector) and magnetic flux density B (2-differential forms) taken along the trajectory of a given particle in a moving medium.

$$\frac{Du}{Dt} = \lim_{k\to 0}\frac{u(x,t)-u(y(x,t-k,t),t-k)}{k} = \frac{\partial u}{\partial t} + u\cdot\nabla u$$

$$\frac{\delta B}{\delta t} = \lim_{k\to 0}\frac{B(x,t)-F^{-1}B(y(x,t-k,t),t-k)}{k} = \frac{\partial B}{\partial t} - \nabla\times(u\times B)$$





Integration on a surface moving with velocity u

Here $y(x,t,s)$ is the flow mapping introduced by velocity field u. F is the Jacobi matrix of the flow map. The last equality makes use of $\nabla\cdot u = 0$ and $\nabla\cdot B = 0$.

$$\frac{d}{dt}\iint_{\Sigma(t)} B\cdot dS = \iint_{\Sigma(t)}\left[\frac{\partial}{\partial t}B + (\nabla\cdot u)B - \nabla\times(u\times B)\right]\cdot dS$$

4. Eulerian-Lagrangian Method

ELM uses an implicit discretization of the material derivative

- results in an *unconditionally stable* scheme
- leads to a *symmetric positive definite* system

One key point is to properly calculate the integrations (of product of two finite element functions from two non-matching grids) on the right-hand side.

After applying finite elements for spatial variables and ELM convection terms: Find $(u_h, p_h, B_h) \in V_h\times Q_h\times E_h$ s.t.

$$k^{-1}(u_h^{n+1}, v_h) + \mu(\nabla u_h^{n+1}, \nabla v_h) + ((\nabla\times B_h^{n+1})\times B_h^{n+1}, v_h) = k^{-1}(u_h^n\circ y(x,t^n,t^{n+1}), v_h), \quad \forall v_h\in V_h$$

$$(\nabla\cdot u_h^{n+1}, q_h) = 0, \quad \forall q_h\in Q_h$$

$$k^{-1}(B_h^{n+1}, e_h) + \eta(\nabla\times B_h^{n+1}, \nabla\times e_h) = k^{-1}(F_{n,n+1}^{-1}B_h^n\circ y(x,t^n,t^{n+1}), e_h), \quad \forall e_h\in E_h$$

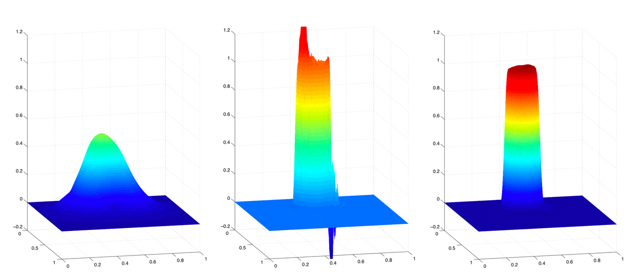
5. Effects of Numerical Integration in ELM

1. Numerical accuracy. With a standard interpolation, ELM for $u_t + bu_x - \epsilon u_{xx} = 0$ leads to the modified equation:

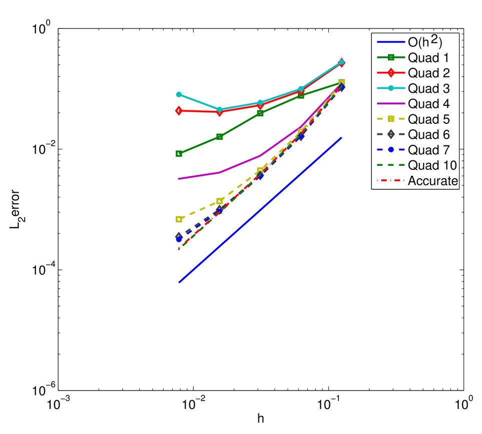
$$u_t + bu_x - (\varepsilon + h)u_{xx} + O(h^2) = 0.$$

But an exact integration leads to the modified equation [6]

$$u_t + bu_x - \varepsilon u_{xx} + M_1 u_{xxxx} + M_2 u_{xxxxx} + O(h^5) = 0.$$

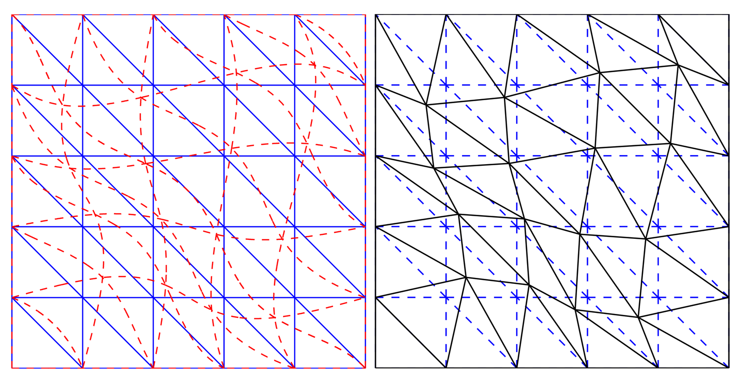


Moving cone with different quadratures

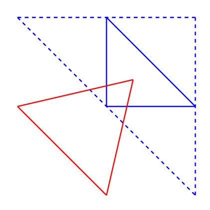


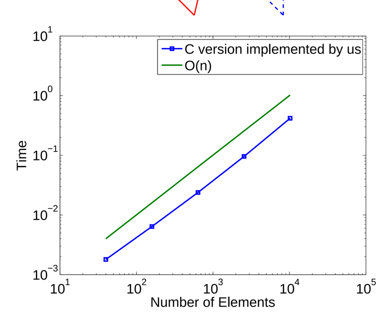
Convergence for B.

2. Accurate (exact) integration. Standard numerical integration does not give much accuracy for the discontinuous edge elements. Accurate evaluation for these integrations in ELM (with optimal complexity) is possible [8]: find the intersection of the grid and deformed grid introduced by the flow mapping and calculate the integration on each intersection exactly.



Approach deformed grid.





One key point: computing the intersection of two elements also gives the information on whether one element's neighbors intersect the other one.

6. Scalable Solver for MHD

ELM for MHD leads to the system (almost linear):

$$\begin{pmatrix} A_1 & N\\ 0 & A_2\end{pmatrix}\begin{pmatrix}(u,p)^T\\ B\end{pmatrix} = \begin{pmatrix}(f,0)^T\\ g\end{pmatrix}$$

1. Stokes-like system. $A_1 = \begin{pmatrix} -\mu\Delta + k^{-1} & (-\mathrm{div})^*\\ -\mathrm{div} & 0\end{pmatrix}$ can be optimally solved for unstructured grids.

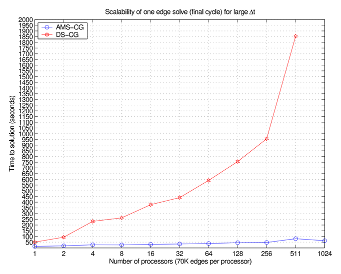
Comparisons of different methods (# iterations/CPU):

DOF	GMRES+T	MINRES+D	SIMPLE
$16K$	21/22	43/45	38/74
$66K$	23/119	48/212	49/449
$262K$	24/504	53/1061	63/2817

Scalability (GMRES+block triangular preconditioner):

DOF	Core	# Iter	CPU (second)
714K	1	24	13.4
5620K	8	24	17.7
44.6M	64	24	34.9
355.M	512	24	50.6

2. Maxwell's equation. HX preconditioner [2, 3, 5] can be used to solve $A_2 = \eta\,\nabla\times\nabla\times + k^{-1}I$ (optimal and scalable for unstructured grids).



7. Ongoing works

1. Applications to benchmark problems (e.g. magnetic reconnection).
2. Combination of grid-adaptation and geometric-algebraic MG
3. Implementations on both CPU (MPI) and GPU.
4. Incompressibility for B_h (also for u_h).

8. References

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