Positivity-preserving high order well-balanced methods for the shallow water equations

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Abstract

Shallow-water equations with a non-flat bottom topography have been widely used to model flows in rivers and coastal areas. An important difficulty arising in these simulations is the appearance of dry areas, and standard numerical methods may fail in the presence of these areas. These equations also have steady-state solutions in which the flux gradients are non-zero but exactly balanced by the source term.

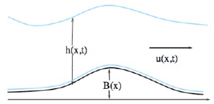
In this work, we propose some recently developed high-order discontinuous Galerkin and weighted essentially non-oscillatory methods, which can preserve the steady-state exactly, and at the same time are positivity preserving without loss of mass conservation. Some numerical tests are performed to verify the positivity, well-balanced property, high-order accuracy, and good resolution for smooth and discontinuous solutions.

Shallow Water Equation

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghb_x \end{cases}$$

h : water height; u : velocity;

b : bottom topography; g : gravitational constant.



It has wide applications in coastal ocean, Tsunami wave propagation, hydraulic engineering and climate.

Numerical Challenge

- **Well-balanced property:**

1. Traditional numerical schemes usually fail to capture the still water steady state

$$u = 0 \qquad \text{and} \qquad h + b = const$$

well and introduce spurious oscillations.

2. Well-balanced methods are specially designed to preserve exactly these steady-state solutions up to machine error with relatively coarse meshes.

- **Appearance of dry areas:**

1. Many applications involve rapidly moving interfaces between wet and dry areas, and standard numerical methods may fail near dry/wet front.
2. Most existing wetting and drying treatments for high order methods focused on post-processing reconstruction of the numerical solution at each time level. It is a challenge to design stable and high order accurate numerical schemes which also have mass conservation.

Main Objective

Develop positivity-preserving high order accurate well-balanced methods for the shallow water equations, which has the key advantage:

- High order accurate
- Well-balanced
- Positivity-preserving with mass conservation
- Good resolution for discontinuous solutions

Well-balanced Schemes

Discontinuous Galerkin methods:

$$\int_{I_j} \partial_t U v dx - \int_{I_j} f(U^n)\partial_x v dx + \widehat{f}^l_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \widehat{f}^r_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = \int_{I_j} s(h^n, b) v dx,$$

WENO methods:

$$\frac{d}{dt}\bar{U}_j(t) + \frac{1}{\Delta x_j}\left(\hat{f}^l_{j+\frac{1}{2}} - \hat{f}^r_{j-\frac{1}{2}}\right) = \frac{1}{\Delta x_j}\int_{I_j} s(h^n, b) dx$$

where the left and right fluxes $\widehat{f}^l_{j+\frac{1}{2}}$ and $\widehat{f}^r_{j-\frac{1}{2}}$ are given by:

$$\widehat{f}^l_{j+\frac{1}{2}} = F(U^{*,-}_{j+\frac{1}{2}}, U^{*,+}_{j+\frac{1}{2}}) + \begin{pmatrix} 0 \\ \frac{g}{2}(h^-_{j+\frac{1}{2}})^2 - \frac{g}{2}(h^{*,-}_{j+\frac{1}{2}})^2 \end{pmatrix}$$

$$\widehat{f}^r_{j-\frac{1}{2}} = F(U^{*,-}_{j-\frac{1}{2}}, U^{*,+}_{j-\frac{1}{2}}) + \begin{pmatrix} 0 \\ \frac{g}{2}(h^+_{j-\frac{1}{2}})^2 - \frac{g}{2}(h^{*,+}_{j-\frac{1}{2}})^2 \end{pmatrix}$$

and

$$h^{*,\pm}_{j+\frac{1}{2}} = \max\left(0, h^{\pm}_{j+\frac{1}{2}} + b^{\pm}_{j+\frac{1}{2}} - \max(b^+_{j+\frac{1}{2}}, b^-_{j+\frac{1}{2}})\right)$$

$$U^{*,\pm}_{j+\frac{1}{2}} = \begin{pmatrix} h^{*,\pm}_{j+\frac{1}{2}} \\ h^{*,\pm}_{j+\frac{1}{2}} u^{\pm}_{j+\frac{1}{2}} \end{pmatrix}.$$

Positivity Preserving limiters

Proposition: Consider the scheme satisfied by the cell averages of the water height in the well-balanced DG/WENO method. If $h^-_{j-\frac{1}{2}}$, $h^+_{j+\frac{1}{2}}$ and $h^n_j(x)$ are all non-negative, then $\overline{h}^{n+1}_j$ is also non-negative under the CFL condition

$$\lambda\alpha \le \widehat{w}_1.$$

Limiter:

To enforce the conditions of this proposition, we introduce the following limiter on the DG polynomial $U^n_j(x) = (h^n_j(x), (hu)^n_j(x))^T$

$$\widetilde{U}^n_j(x) = \theta\left(U^n_j(x) - \overline{U}^n_j\right) + \overline{U}^n_j, \qquad \theta = \min\left\{1, \frac{\overline{h}^n_j}{\overline{h}^n_j - m_j}\right\},$$

with

$$m_j = \min_{x \in I_j} h^n_j(x).$$

This limiter does not destroy the high order accuracy, and preserves the local conservation of *h* and *hu*.

Implementation of limiter:

Replace

$$m_j = \min_{x\in I_j} h^n_j(x),$$

by

$$m_j = \min(h^+_{j-\frac{1}{2}}, h^-_{j+\frac{1}{2}}, \xi_j). \quad \xi_j = \frac{\overline{h}^n_j - \hat{w}_1 h^+_{j-\frac{1}{2}} - \hat{w}_N h^-_{j+\frac{1}{2}}}{1 - \hat{w}_1 - \hat{w}_N}.$$

Algorithm flowchart

- Evaluate m_j.
- Use the positivity-preserving limiter to compute $\widetilde{U}^n_j(x)$.
- Compute the well-balanced fluxes from $\widetilde{U}^n_j(x)$.
- Use $\widetilde{U}^n_j(x)$ instead of $U^n_j(x)$ in the scheme with the corresponding CFL condition.

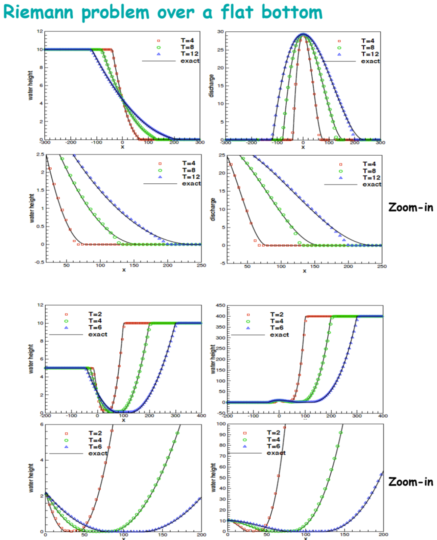
The results can be extended to the two dimensional problems.

Numerical Results

Accuracy test

Number of cells	CFL	h L^1 error	order	hu L^1 error	order	hv L^1 error	order
25×25	0.6	9.39E−03		3.01E−02		7.38E−02	
50×50	0.6	1.27E−03	2.89	2.37E−03	3.67	1.03E−03	2.84
100×100	0.5	1.03E−04	3.61	1.41E−04	4.07	8.39E−04	3.61
200×200	0.4	5.01E−06	4.37	6.19E−06	4.51	4.06E−05	4.37
400×400	0.3	1.87E−07	4.75	2.27E−07	4.77	1.50E−06	4.76
800×800	0.2	5.83E−09	5.00	7.50E−09	4.92	4.73E−08	4.98

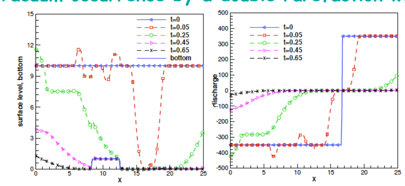
Riemann problem over a flat bottom



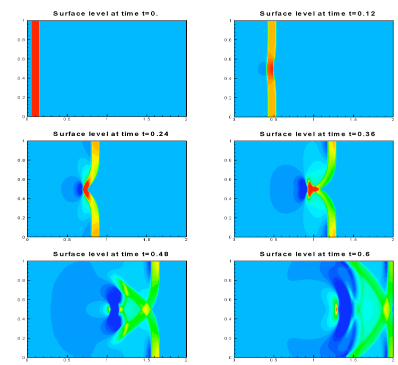
Test for well-balanced property

precision	L^1 error h	L^1 error hu	L^∞ error h	L^∞ error hu
single	2.89E-07	1.14E-07	5.81E-07	4.20E-07
double	7.16E-16	1.94E-16	1.11E-15	1.42E-15

Vacuum occurrence by a double rarefaction wave



Small perturbation test



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