Construction and Properties of Coarse Discretization Spaces by AMGe

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Abstract: Several approaches to construct AMG coarse finite element discretization spaces are presented based on element agglomeration AMG (or AMGe) as well as smoothed aggregation AMG (or SA AMG). The spaces can be used both for building multilevel iterative methods as well as upscaling (discretization) tool. This is illustrated on diffusion equation with large jumps in coefficient in primal and mixed formulation.

Diffusion equation with coefficient not resolved by a coarse grid .

$$-div(k(x)\nabla p(x)) = f(x) \text{ in } \Omega;\ p(x) = g(x) \text{ on } \partial\Omega.$$

Mixed form: Find u and p such that:

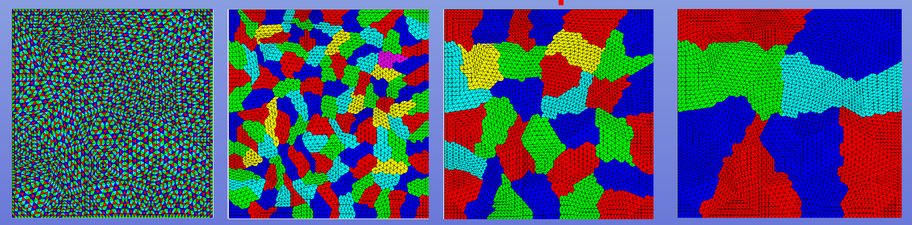
$$(k^{-1}u, v) + (p, div\, v) = 0, \qquad \text{for all } v \in H(div, \Omega),$$

$$(div\, u, q) \qquad = -(f, q), \text{ for all } q \in L_2(\Omega).$$

Mesh and Elements

- The coefficient is constant on each fine element
- Fine elements are combined in agglomerated elements (AE)
- Local stiffness matrices (A_T) are assembled for each AE
- For SA AMG, aggregates are generated corresponding to AEs.

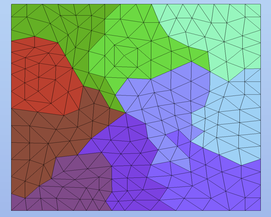
Smooth coefficient test[4]: spectral AMGe



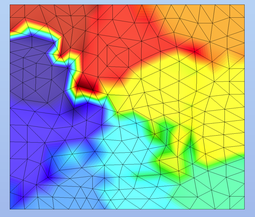
level	# elements	# dofs	# nnz	$\|\|u_h - u_H\|\|_0$	$\|\|u_h - u_H\|\|$energy
0	495538	988820	3464252		
3	16384	89917	1998045	3.171225e-04	1.366769e-02
4	4096	24270	574950	5.690525e-04	1.820443e-02
5	1024	6549	160269	8.708961e-04	2.245570e-02
6	256	1875	48361	1.132020e-03	2.553637e-02
7	64	701	21367	1.464650e-03	2.922599e-02
8	16	306	11078	1.538692e-03	3.001590e-02

Smoothed Aggregation Spectral Element-Agglomeration AMG[1] (TL and ML results)

$$-div\, k(x)\nabla p(x) = 1 \text{ in } \Omega,\ p(x) = 0 \text{ on } \partial\Omega.$$



Agglomerated elements



Aggregates

c	# coarse grid dofs	n_{it}	ϱ
-12	2,729	53	0.74
-9	2,730	53	0.75
-6	2,730	51	0.74
-3	2,729	26	0.60
0	2,660	22	0.60
3	2,728	24	0.60
6	2,744	36	0.65
9	2,744	46	0.70
12	2,744	46	0.70

TABLE 1. Convergence factor (ϱ) and number of iterations (n_{it}) for the two–level SA spectral AMGe with $\nu = 6$ (degree of polynomial smoother $3\nu+1 = 19$). The fine (triangular) mesh is fixed with 465,712 elements and 233,499 dofs. The coarse mesh consists of 1,330 elements. The jump in the PDE coefficient is 10^c.

c	# coarse grid dofs	n_{it}^{TL}	ϱ_{TL}	n_{it}^{ML}	ϱ_{ML}
-12	11,191	57	0.73	77	0.79
-9	11,191	56	0.73	77	0.79
-6	11,191	46	0.70	67	0.78
-3	11,193	21	0.53	39	0.71
0	11,073	19	0.54	33	0.72
3	11,199	21	0.53	43	0.73
6	11,199	36	0.64	44	0.71
9	11,199	39	0.65	49	0.69
12	11,199	39	0.65	58	0.71

TABLE 2. Convergence factor (ϱ) and number of iterations (n_{it}) for both two-level (TL) and the multilevel (ML) SA spectral AMGe with $\nu = 6$ (degree of polynomial smoother $3\nu+1 = 19$) at the finest level. At coarser level (in the ML case) block Gauss–Seidel smoother is used with blocks corresponding to the respective aggregates. The fine (triangular) mesh is fixed with 465,712 elements and 233,499 dofs. The coarse mesh consists of 5,250 elements. The jump in the PDE coefficient is 10^c.

# fine grid dofs	# fine grid elements	# dofs at 1st coarse level	# elements at 1st coarse level	n_{it}^{ML}	ϱ_{ML}	n_{it}^{TL}	ϱ_{TL}	operator complexity
40,377	80,192	1,237	570	17	0.36	13	0.26	1.13
100,613	200,416	3,410	1,596	29	0.55	19	0.49	1.14
233,499	465,712	11,199	5,250	58	0.71	39	0.65	1.19
578,017	1,153,792	43,026	20,860	84	0.80	38	0.63	1.29

TABLE 3. Convergence factor (ϱ) for the two–level and multilevel SA spectral AMGe method with polynomial smoother at the finest level ($\nu = 6$) and block Gauss–Seidel at coarser levels with blocks corresponding to aggregates. Fixed jump of 10^{12} in the PDE coefficient and variable fine-grid mesh. The last column shows the operator complexity of the multilevel method; namely, the sum of the non-zero entries of all level matrices divided by the number of non–zero entries of the finest-level matrix.

Mesh and PDE coefficient

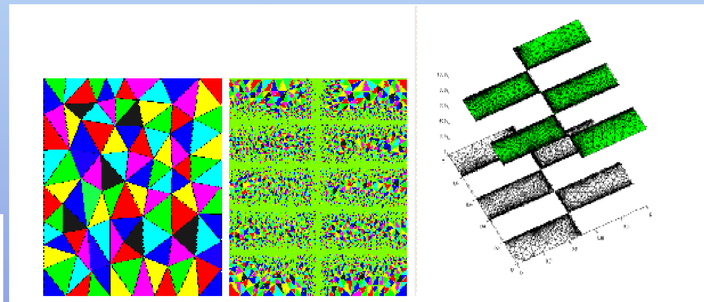


Fig. 1. Initial unstructured mesh (left) that does not resolve the discontinuous coefficient, its refinement (middle) that resolves the high-contrast discontinuous coefficient, and the actual distribution of the coefficient values 1 and 10^c, $c = 3$ (right).

To ensure approximation property: $H^{-2}\|v - v_H\|_0^2 + a(v - v_H, v - v_H) \le Ca(v, v).$

Solve local AE eigen-problems to construct local P and block-diagonal global tentative $\hat{P}$

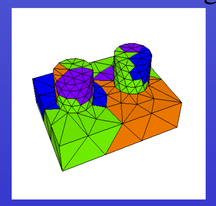
$$A_T \mathbf{q}_k = \lambda_k D_T \mathbf{q}_k,\ k = 1, \ldots, n_T.$$

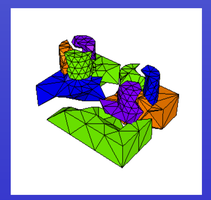
Use SA polynomial to form global P

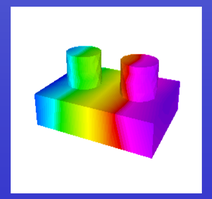
$$s_\nu(t) = (-1)^\nu \frac{1}{2\nu+1} \frac{T_{2\nu+1}(\sqrt{t})}{\sqrt{t}}, \quad S = s_\nu(D^{-1}A) \quad P = S\hat{P}.$$

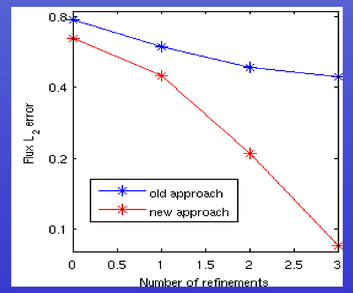
Coarse Raviart-Thomas Spaces with Approximation Properties

Main idea of the method: construct coarse space that interpolates vector constants (gradients of linear functions, like in the third figure below) *exactly.*









Comparison of approximation properties of new method [2] versus [3].

References

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[2] Ilya Lashuk, P.S. Vassilevski, LLNL Report #LLNL-JRNL-485771, June 6, 2011

[3] J.E. Pasciak and P.S. Vassilevski, "Exact de Rham Sequences of Spaces Defined on Macro-elements in Two and Three Spatial Dimensions," SIAM Journal on Scientific Computing **30** (2008), pp. 2427-2446.

[4] P.S. Vassilevski, "Coarse Spaces by Algebraic Multigrid: Multigrid Convergence and Upscaling Error Estimates,"

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