High resolution solutions for shock formation in transonic flow problems

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Introduction

Shock waves appear generically in solutions of transonic flow problems when rarefaction waves reflect off of sonic lines. Some examples of this occur in

(1) Transonic flow over an airfoil. When the flow approaching an airfoil has a high subsonic velocity, the expansion wave created by the airfoil profile creates a local supersonic bubble over the airfoil. This expansion wave reflects off the sonic line as a compression which typically forms a terminating shock (cf [1]).

(2) Supersonic flow of a gas hitting the corner of an expanding duct. As shown in [3], the rarefaction wave generated at the corner reflects off a sonic line, and again a shock forms.

(3) Weak shock reflection off thin wedges, known as Guderley Mach reflection. Numerical solutions in [2], [4] show that, not just a single shock, but a sequence (perhaps infinite) of shocks are formed by the reflection of expansion waves off a sonic line.

In each of the example problems above, the question arises: Where does the shock form?

Whether a shock forms on the sonic line or inside the supersonic region is an open question – no mathematical proof exists.

Also, until recently, numerical evidence supporting either possibility has been inconclusive. Many numerical solutions of the transonic airfoil problem have appeared in the literature. These solutions appear to show that the shock begins on, or very close to, the sonic line, but cannot distinguish between these possibilities.

Our aim: determine where shock formation occurs. We use high resolution finite difference methods and local grid refinement, and solve problems which describe Examples (1) and (2) above.

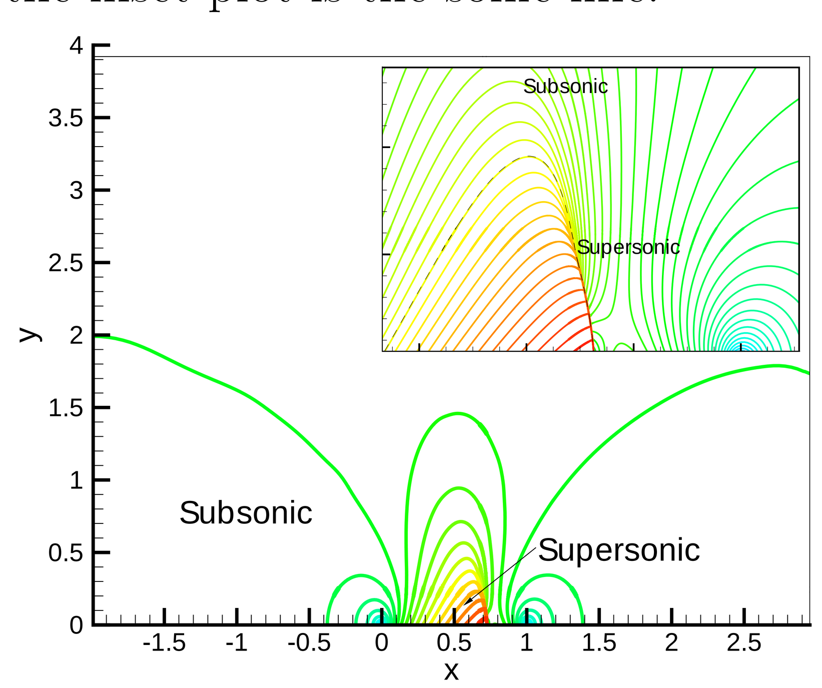
The transonic airfoil problem

We solve the classical problem of flow over a thin airfoil with an incoming freestream Mach number which is slightly subsonic. The governing equation is the steady transonic small disturbance equation (TSDE), which can be written

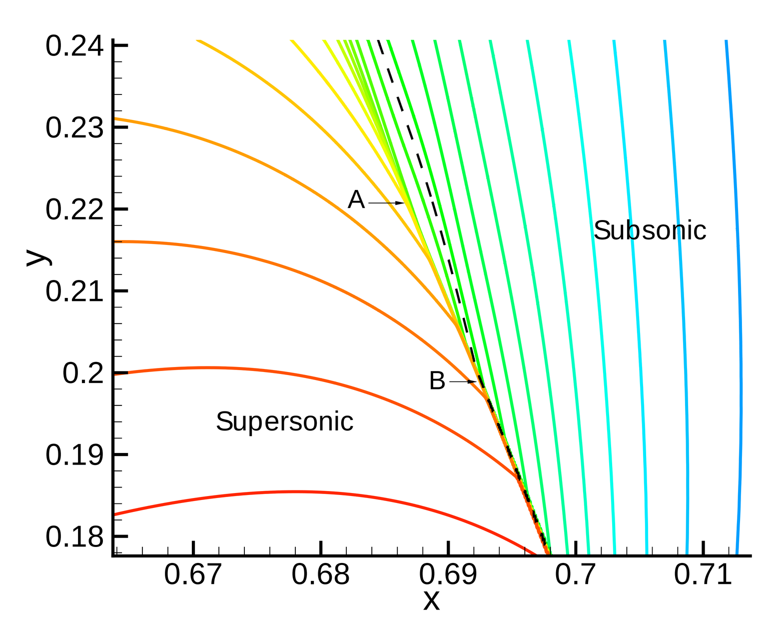
$$\left[\left(1-M_\infty^2\right)\phi_x - \frac{\gamma+1}{2}M_\infty^2\,\phi_x^2\right]_x + \phi_{yy} = 0.$$

Here, $\phi(x,y)$ is the disturbance velocity potential, where $u=\phi_x$ and $v=\phi_y$ are the disturbance velocity components in the x and y directions; M_∞ is the freestream Mach number; and γ is the ratio of specific heats. We use a high resolution finite difference scheme (see [1]).

The following figure shows a numerical solution depicting the flowfield above a transonic airfoil at $M_\infty = 0.8$ and zero incidence. Contours of u-velocity are shown; the inset plot shows an enlargement of the supersonic bubble region. The dashed line in the inset plot is the sonic line.



The shock appears to begin on, or very close to, the sonic line. To clarify this, the figure below depicts the solution in a small region near the shock formation point, where extreme local grid refinement was used:



As shown, the shock forms in the supersonic region.

The expanding duct problem

A problem which describes a slightly supersonic flow hitting the corner of an expanding duct at $t=0$ was introduced in [3]. It consists of the unsteady TSDE,

$$u_t + \left(\frac{1}{2}u^2\right)_x + v_y = 0,$$
$$u_y - v_x = 0,$$

in the half space $y>0$, together with initial data given by

$$(u,v) = \begin{cases}(0,0) & \text{if } x > -by,\\ (-1,-b) & \text{if } x < -by,\end{cases}$$

and the no-flow boundary condition on $y=0$,

$$v(x,0,t) = 0.$$

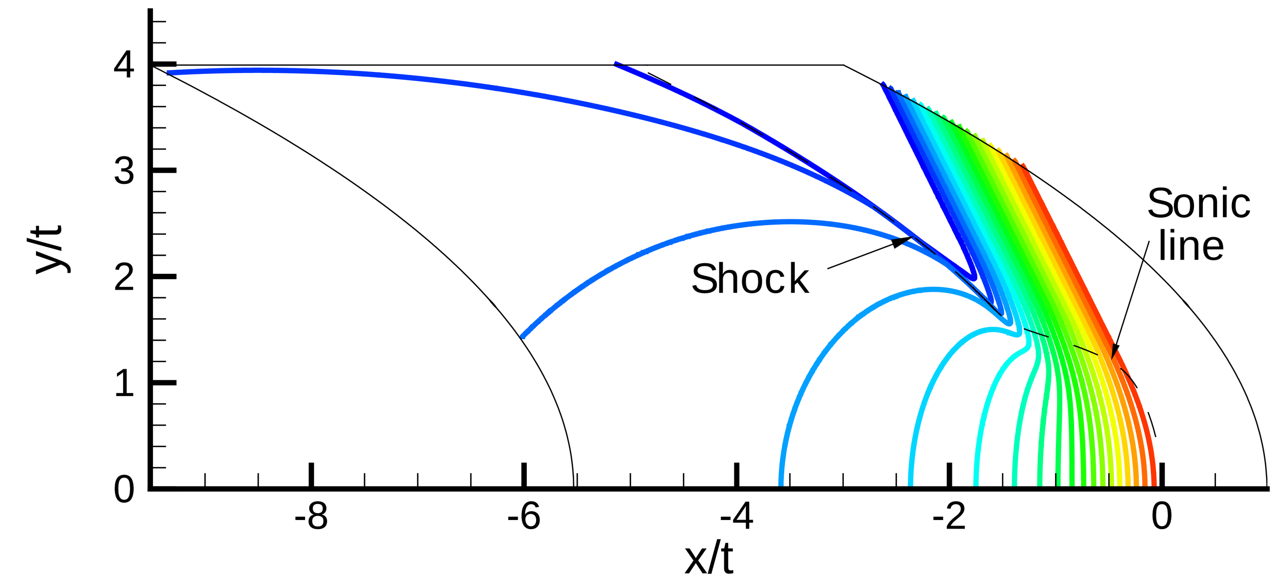
Again u and v are velocity components in the x and y directions, and the parameter $b>0$.

This problem is self-similar, so the solution depends only on the similarity variables $\xi = x/t, \eta = y/t$. Writing the unsteady UTSDE in terms of ξ and η, we get

$$-\xi u_\xi - \eta u_\eta + \left(\frac{1}{2}u^2\right)_\xi + v_\eta = 0,$$
$$u_\eta - v_\xi = 0.$$

This equation is hyperbolic when $u < \xi + \eta^2/4$, corresponding to supersonic flow, and elliptic when $u > \xi + \eta^2/4$, corresponding to subsonic flow. We solve it numerically as described in [1].

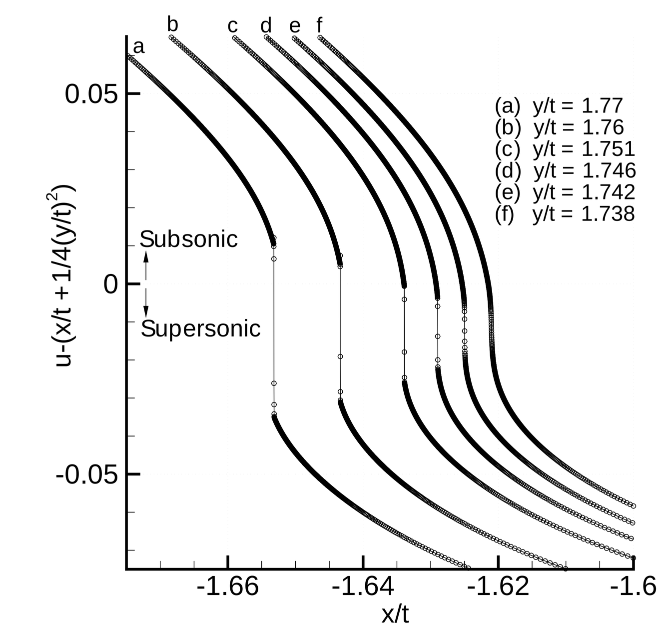
This figure shows a numerical solution of the expanding duct problem (u contours shown). An expansion wave reflects off the sonic line and a shock wave is formed. The boundaries are curved because of the self-similar variables used:



From the figure, the shock appears to form on or very close to the sonic line, at $y/t \approx 1.8$.

The expanding duct (concl.)

To determine where the shock forms, in the next figure we plot profiles of the function $u - \xi + \eta^2/4$, taken horizontally across the shock in a small region close to the shock formation point:



These profiles show that the shock (represented by a jump) forms at approximately $y/t = 1.742$. The jump is from supersonic to supersonic values of $u - \xi + \eta^2/4$, so the shock forms in the supersonic region.

Main result

We find that the shock forms strictly in the supersonic region, due to coalescence of compression waves reflected from a sonic line. This is the first time that this has been directly observed.

References

[1] A. M. Tesdall, High resolution solutions for the supersonic formation of shocks in transonic flow, J. Hyperbolic Differ. Equ. volume 8 (2011), 485–506.

[2] A. M. Tesdall and J. K. Hunter, Self-similar solutions for weak shock reflection, SIAM J. Appl. Math. volume 63 (2002), 42–61.

[3] A. M. Tesdall and B. L. Keyfitz, A continuous, two-way free boundary in the unsteady transonic small disturbance equations, J. Hyperbolic Differ. Equ. volume 7 (2010), 317–338.

[4] A. M. Tesdall, R. Sanders, and B. L. Keyfitz, Self-similar solutions for the triple point paradox in gasdynamics, SIAM J. Appl. Math. volume 68 (2008), 1360–1377.