Slope Limiters for Finite Volume Schemes on Non-coordinate-aligned Meshes

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Motivation

Realistic engineering geometries can be very complex. Cartesian embedded boundary methods are an automatic and robust way to handle complex geometries.

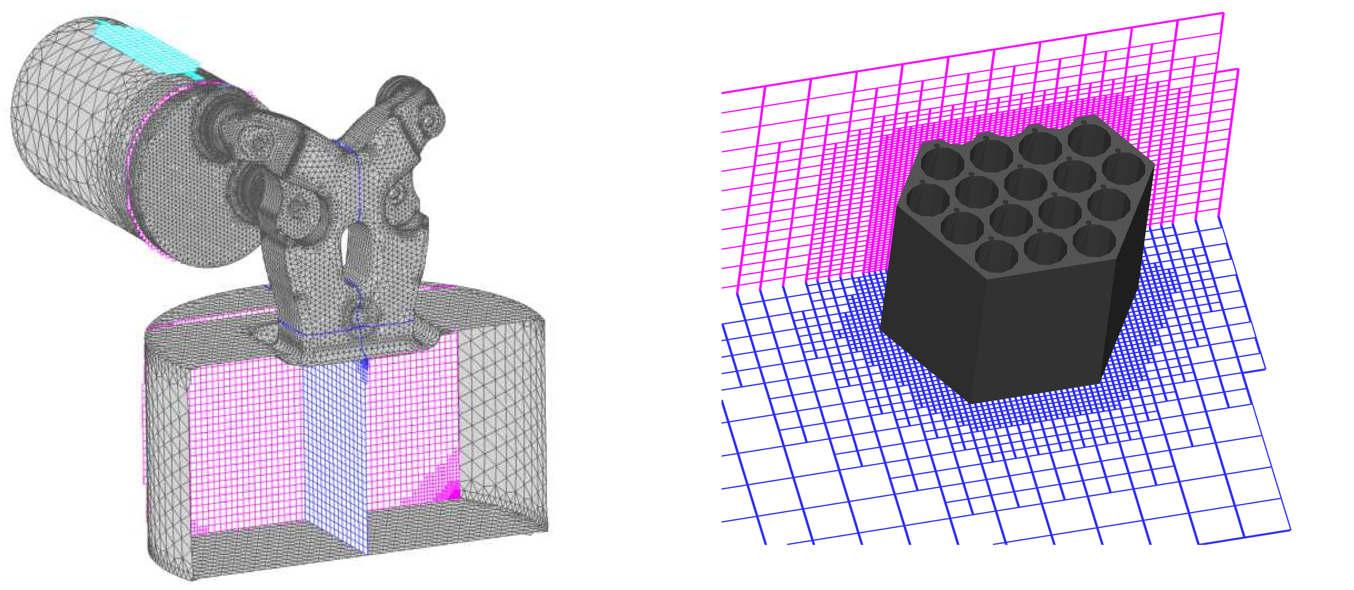

Figure 1: Illustration of Cartesian embedded boundary meshes for an engine (left) and fuel rods (right).

One major difficulty of this approach is the handling of **'cut cells'**. These cells arise when a solid object is cut out of a Cartesian background grid (see Figure to the right).
2nd order Gudonov-type methods require the reconstruction of a linear (limited) polynomial which is non-trivial for non-coodinate-aligned meshes.

Goal of this work: Construction of slope limiters on cut cells

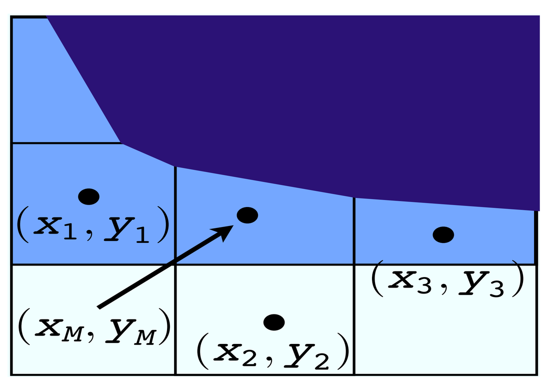

Figure 2: Light blue cells = cut cells.

Difficulties:

1. **Centroids of grid cells not aligned with coordinate axes**
2. **Want the slope limiter to guarantee monotonicity constraints**

Slope limiting for cut cells

Least-Squares Estimate

Let u be the value at the centroid. To obtain an estimate for the gradient (D_x, D_y) on cell M (which has 3 edge neighbors) we solve the least-squares problem:

$$\begin{bmatrix} x_1 - x_M & y_1 - y_M \\ x_2 - x_M & y_2 - y_M \\ x_3 - x_M & y_3 - y_M \end{bmatrix} \begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} u_1 - u_M \\ u_2 - u_M \\ u_3 - u_M \end{bmatrix}.$$

Standard Scalar Limiting

Reduce D_x and D_y using a **scalar** factor:

$$u(x,y) = u_M + \phi \begin{bmatrix} D_x \\ D_y \end{bmatrix} \cdot \begin{bmatrix} x - x_M \\ y - y_M \end{bmatrix}.$$

Choose ϕ to prevent overshoot when reconstructing to neighbors: for $i \in \{\text{neighbors}\}$ ensure (if $u_M \le u_i$)

$$u_M + \phi \begin{bmatrix} D_x (x_i - x_M) \\ D_y (y_i - y_M) \end{bmatrix} \le u_i \, .$$

New LP Limiting

Limit **each** coordinate **direction separately**:

$$u(x,y) = u_M + \begin{bmatrix} \phi_x D_x \\ \phi_y D_y \end{bmatrix} \cdot \begin{bmatrix} x - x_M \\ y - y_M \end{bmatrix}.$$

Approach based on the work by *Berger, Aftosmis & Murman* [BAM05]. This leads to the following

Linear Program:

$$\min_{\phi_x, \phi_y} \quad |D_x - \phi_x D_x| + |D_y - \phi_y D_y| = -\phi_x |D_x| - \phi_y |D_y| + \text{const}$$

subject to

$$0 \le \phi_x, \phi_y \le 1$$

and for each neighbor (assume $u_M \le u_i$):

$$u_M + \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} \cdot \begin{bmatrix} D_x (x_i - x_M) \\ D_y (y_i - y_M) \end{bmatrix} \le u_i \qquad \text{(don't overshoot)},$$

$$\begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} \cdot \begin{bmatrix} D_x (x_i - x_M) \\ D_y (y_i - y_M) \end{bmatrix} \ge 0 \qquad \text{(don't violate descent direction)}.$$

To *guarantee positivity* (e.g., for density), add constraint for boundary edge mid-point (x_{bry}, y_{bry}):

$$u_M + \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} \cdot \begin{bmatrix} D_x (x_{bry} - x_M) \\ D_y (y_{bry} - y_M) \end{bmatrix} \ge 0.$$

Solving the Linear Program

Q: How do we solve LP? **A:** Use *all-inequality* **Simplex method**. (Work in every iteration proportional to number of variables (i.e., two in 2d) – for standard Simplex method work proportional to number of constraints.) Note that $(0,0)$ (the zero-gradient) is basic feasible point of LP.

Family of LP limiters

Varying the constraints gives rise to a whole **family of LP limiters**. Most scalar limiting versions have corresponding limiter in the LP family. Depending on

- the choice of the unlimited gradient
- and the formulation of the constraints

the resulting LP limiter is more suitable for problems with strong shocks or overall smooth behavior.

Numerical results

Supersonic Vortex Test

We first consider the inviscid, isentropic, supersonic flow of a compressible fluid between concentric arcs as presented in [AGT94]. Since the flow is **shock-free** and there exists an exact, closed form, **analytic solution**, this steady-state test allows to measure the accuracy of our new limiter. For better comparison we only limit cut cells.

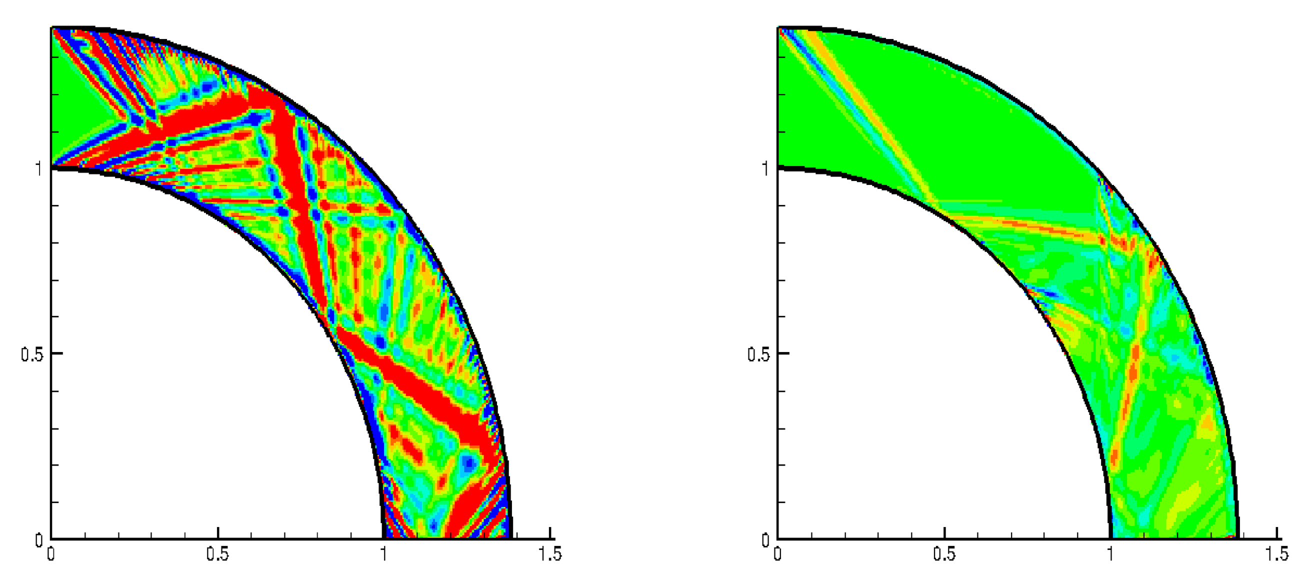

Figure 3: Error in density for *scalar* (left) and *LP* limiter (right). In both cases, the constraints shown on this poster were used.

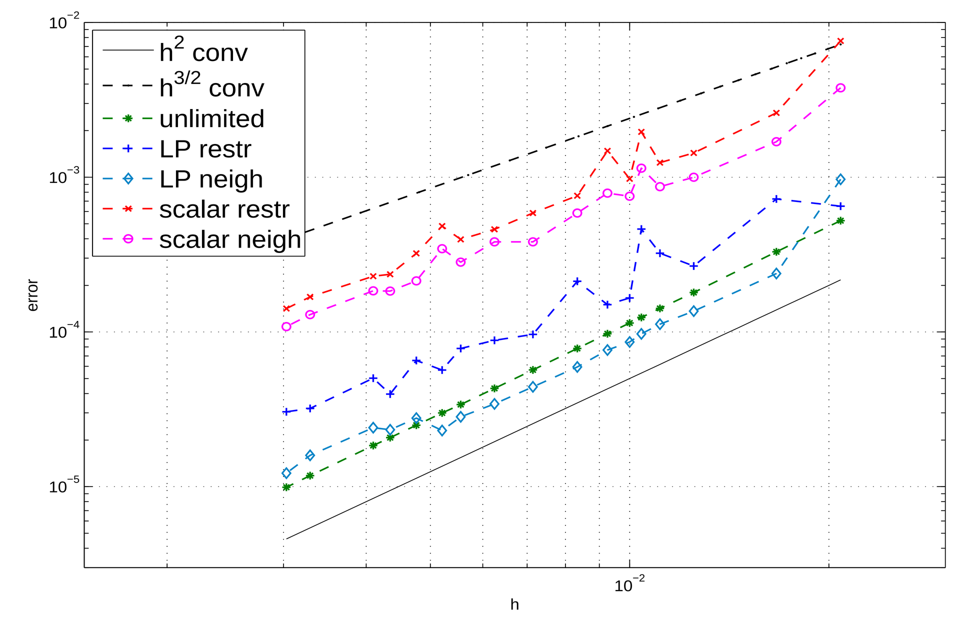

Figure 4: Convergence of density error in L^1-norm. 'LP restr' and 'scalar restr' use constraints shown on this poster. 'LP neigh' and 'scalar neigh' use min/max over neighbors for limiting.

Cylinder Test

Our LP limiter works well in the **presence of discontinuities** as the following test shows: Initially a Mach 2 shock is located at $x = 0.2$. The state in front of the shock is $\rho = 1.4$, $u = v = 0$, $p = 1$.

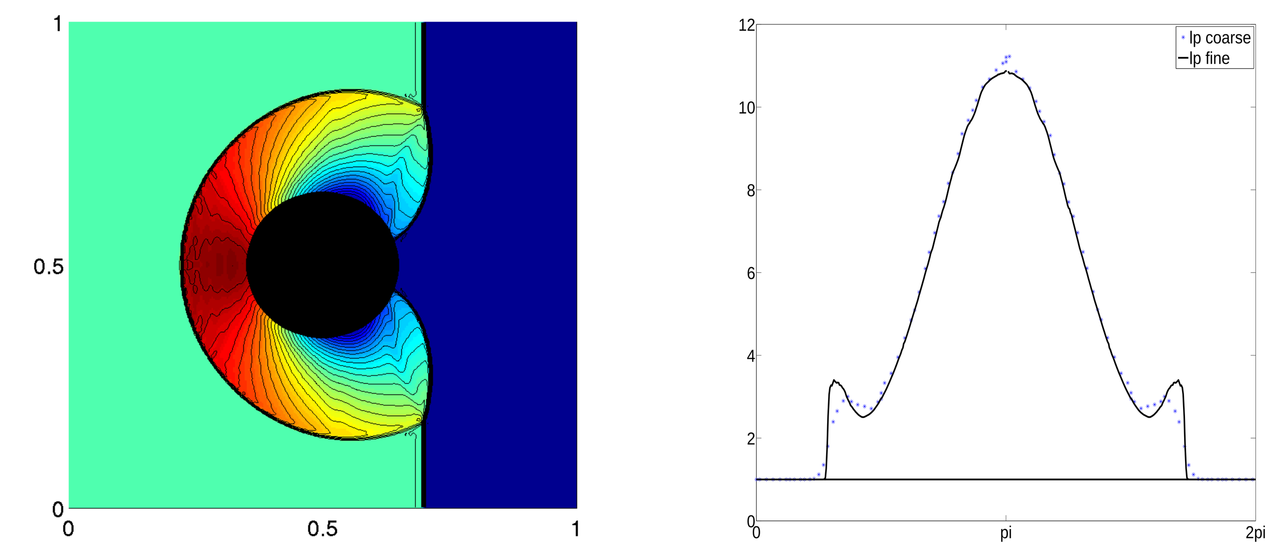

Figure 5: Cylinder test using restrictive LP limiter: left: Density at $t = 0.25$. right: Pressure along the boundary of the cylinder at $t = 0.20$ (coarse and fine grid).

Triangular grids: In [Hub98], Hubbard suggested formulating the limiter as an optimization problem (but didn't give an algorithm for its solution). We tested our LP limiter on structured triangles. For a restrictive choice of constraints, our LP limiter was usually a factor of 2-4 better than the corresponding scalar limiter. If constraints are loosened (and therefore the main error is caused by cutting extrema), LP and scalar do equally well. We expect more of a difference for unstructured triangular grids.

Cost of LP limiter: In our tests the Simplex algorithm needed an average of 2-3 iterations to solve the LP. In every iteration, two 2-by-2 systems are inverted ⇒ LP is roughly 2-3 times as expensive as scalar ⇒ cost is higher but results are more accurate

Conclusions/Future Work

Our new LP limiter

- is more accurate than the scalar limiter.
- handles discontinuities well.
- is able to deal with specific requests such as positivity at boundary edge midpoint.

We are currently testing the LP limiter family on 3d problems.

References

[AGT94] M. Aftosmis, D. Gaitonde, and T. Sean Tavares. On the accuracy, stability and monotonicity of various reconstruction algorithms for unstructured meshes. In *32nd AIAA Aerospace Sciences Meeting, Reno, NV*, 1994. Paper AIAA 94-0415.

[BAM05] M. Berger, M. J. Aftosmis, and S. M. Murman. Analysis of slope limiters on irregular grids. In *43rd AIAA Aerospace Sciences Meeting, Reno, NV*, 2005. Paper AIAA 2005-0490.

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