Shock Formation at the Sonic Line

Analysis of Self-Similar Solutions of Multidimensional Conservation Laws

Barbara Lee Keyfitz, Department of Mathematics, The Ohio State University, Columbus, OH 43210

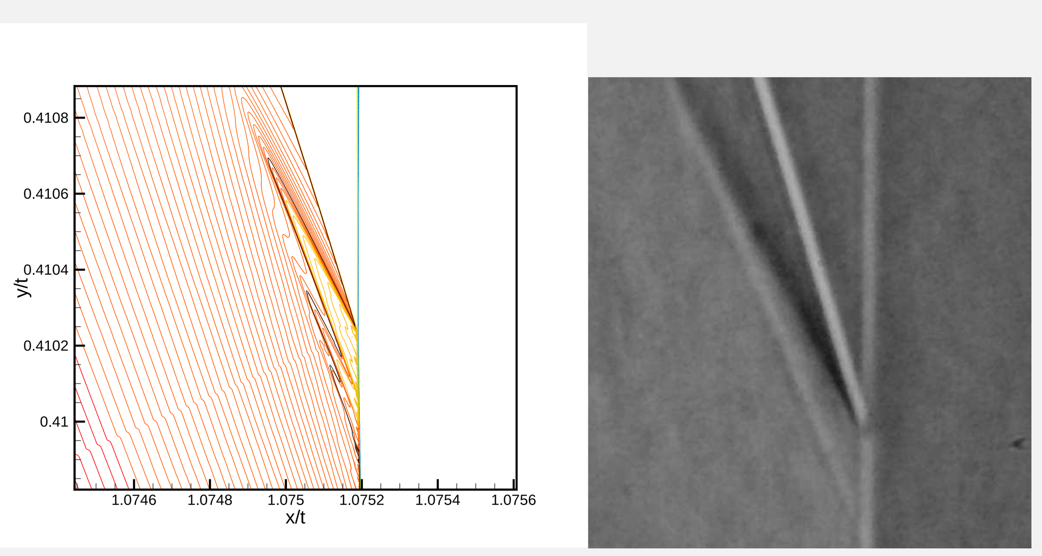
Background: Multidimensional Conservation Laws

- Attempt to develop theory, beginning about 1990
- Following
  - Long history of computational results
  - Successes in establishing well-posedness for 1-D systems
- Several groups:
  - Computational: Bell, Colella, Henderson, et al [4]
    - Tabak and Rosales [13]
    - Chen, Feldman et al [3]
    - Hunter and Brio [7]
    - Elling and Liu [5]
    - Čanić, K et al [1, 9]

Self-Similar Problems aka "Two-Dimensional Riemann Problems"

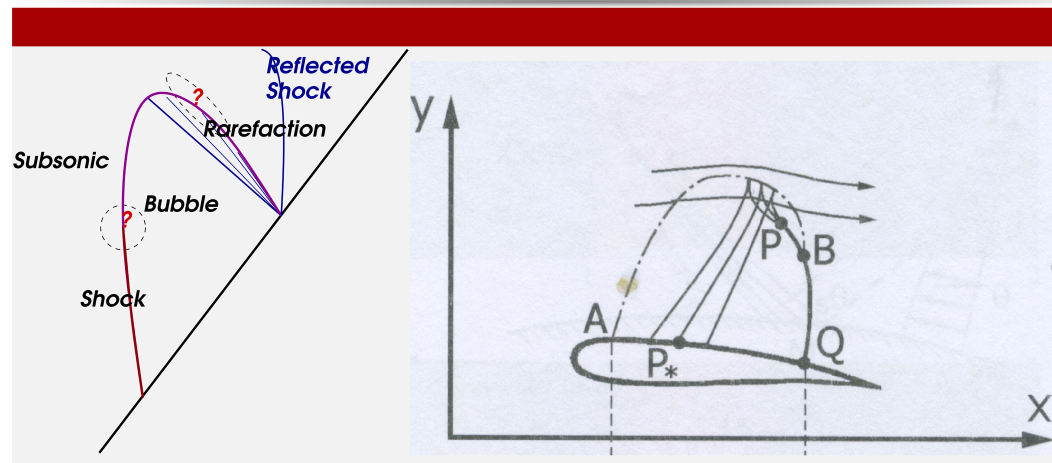
- System $U_t + F(U)_x + G(U)_y = 0$ becomes
$$-\xi U_\xi - \eta U_\eta + F(U)_\xi + G(U)_\eta = 0$$
with $\xi = \frac{x}{t}$ $\qquad \eta = \frac{y}{t}$
- Analogy with steady flow, $F_x + G_y = 0$
- Interest stems from some benchmark problems
- Motivates new mathematical techniques

Guderley Mach Reflection



Context
- "Weak Shock Reflection" (small wedge angle, near-sonic Mach number)
- von Neumann paradox
- New phenomenon predicted by Guderley [6]
- Noted by Čanić & K
- Discovered numerically by Tesdall and Hunter [14]
- Confirmed in experiments by Skews et al. [12]
- TSK simulation motivated Skews experiment [15]

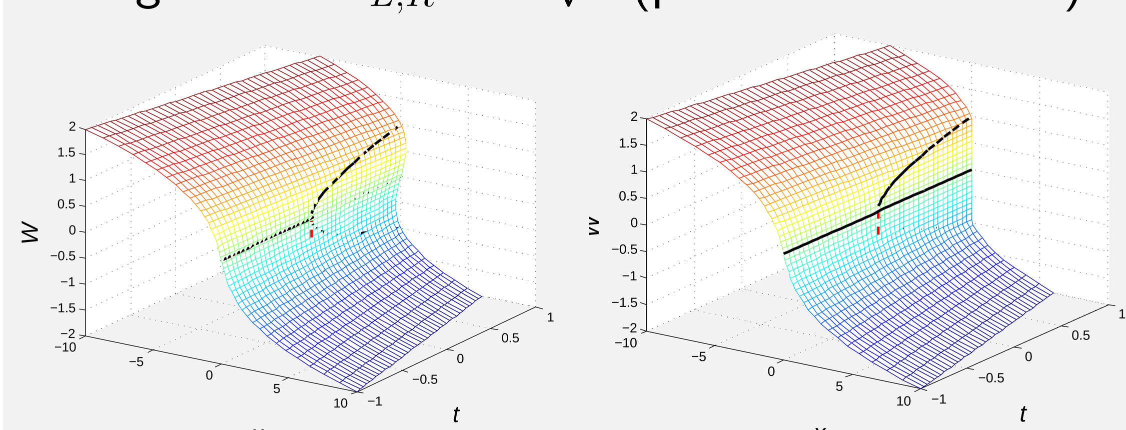
Anatomy of a Supersonic Bubble



- Shock formed by compression at sonic line
- Comparison: Trailing Edge of Transonic Airfoil Kuz'min: Shock forms inside supersonic region [11]

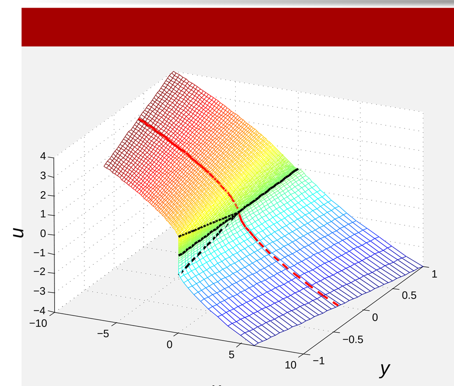
Hyperbolic Shock Formation: The Cusp Singularity

Standard $3/2$ cusp surface $W = W(x,t)$ is given by
$$W^3 - tW + x = 0$$
Solves Burgers equation $W_t + WW_x = 0$
- Formation point $(0,0)$: $W_x(x,0) = -\frac{1}{3}x^{-2/3}$
- Along shock $W_{L,R} = \pm\sqrt{t}$ (parabolic X-section)



- Typical of shock formation at a hyperbolic point
- Valid for genuinely nonlinear flux
- Valid for hyperbolic systems

Sonic Shock Formation: Distorted Cusp



TSDE $\begin{cases} uu_x + v_y = 0 \\ v_x - u_y = 0 \end{cases}$

Hyperbolic $u < 0$:

$\lambda = -\sqrt{-u}$, $\rho = \sqrt{-u}$

RI: $w_y + (w/2)^{1/3} w_x = 0$

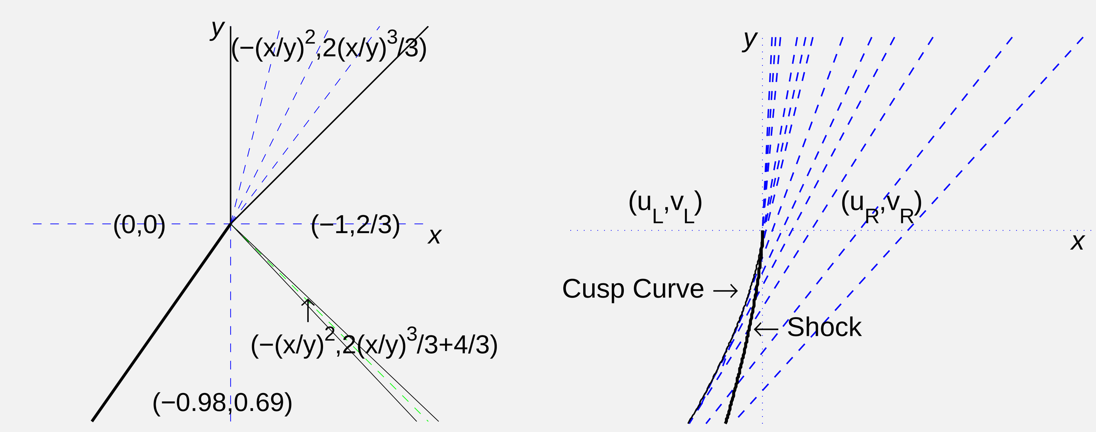
$W = w^{1/3}$ & $W_y + WW_x = 0$

$u(x,y) = -W^2(x,y)$:
$$u_x(x,0) = -\frac{1}{3}\left(\frac{x}{2}\right)^{-1/3} \qquad u(0,y) = -\frac{|y|}{2}$$

Idea: $\rho = \sqrt{-u}$ behaves like a cusp

Construction: TSDE Sonic Shock

Direct two compression waves towards the origin; hold downstream flow exactly sonic

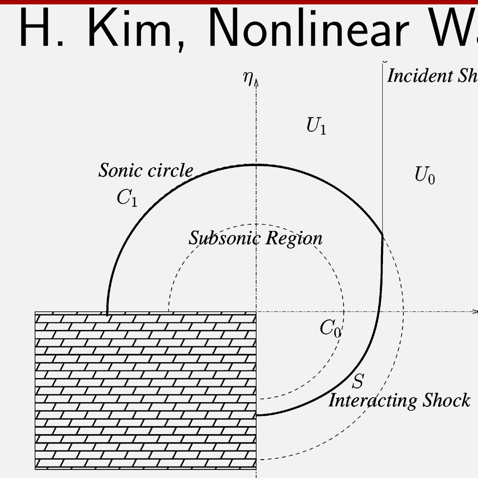


Infinitesimal version (conjectured):

Downstream $u_L = x^2 u_1(x,y) \quad v_L = x^3 v_1(x,y)$
$$(2u_1 + xu_{1x})u_1 + v_{1y} = 0$$
$$3v_1 + xv_{1x} - u_{1y} = 0$$

'Subsonic Compression'

E. H. Kim, Nonlinear Wave System [10]:

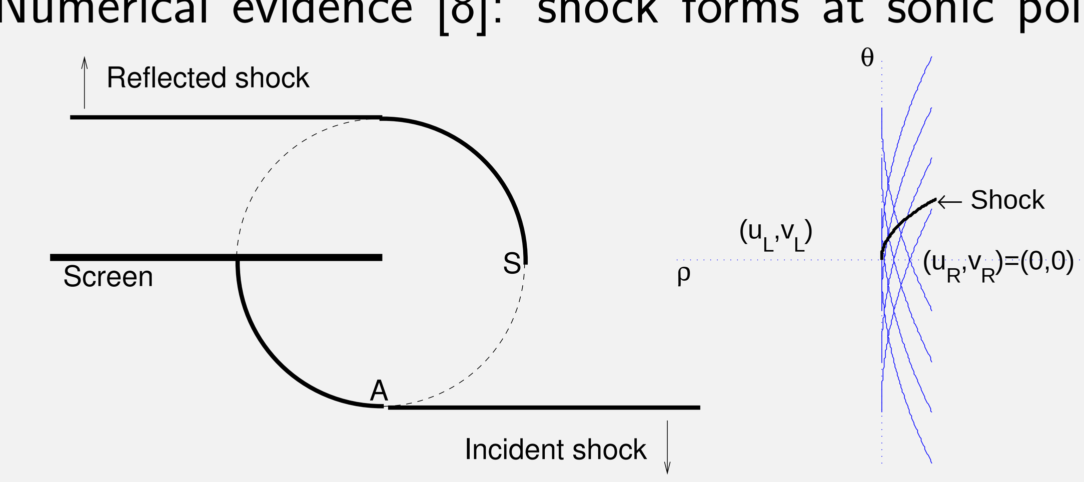


$$p = u - u_R$$
$$q = v - v_R$$

$$(u_R + p)p_x + u_{Rx}p + q_y = -u_R u_{Rx} - v_{Ry}$$
$$q_x - p_y = -v_{Rx} + u_{Ry}$$

Self-Similar Diffraction (Hunter-Tesdall)

Numerical evidence [8]: shock forms at sonic point



UTSD model $\begin{cases} w_\theta + (\rho + u)u_\rho - u/2 = 0 \\ u_\theta - w_\rho = 0 \end{cases}$

Asymptotics:
- $(u_R, v_R) = (0,0)$
- $6b^2 + b_{\theta\theta} + \ldots = 0$
- $u_L = -\frac{\rho}{2} + \rho^2 b(\rho,\theta)$
- $[u] = \frac{3\theta^2}{2}$ along shock

Conclusions

Bad conjecture: Shocks don't form at sonic points
- Sonic shock Riemann solutions exist
- Infinitesimal steady sonic shock generation
  - requires compression wave in the hyperbolic region
  - is unstable to small perturbations of the data
  - differs from strictly hyperbolic shock formation
- Quasi-steady (self-similar) problems can generate in the subsonic compression
- Quasi-steady shocks have a different form (no derivative blow-up)
- Shock growth rate is related to nonlinearity in the characteristic speeds and to compression rate of characteristic curves

Current project: Clarify these preliminary results

Acknowledgements

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