A 2-edge-connected spanning subgraph problem: Robert Carr, Ojas Parekh, Sandia Labs

Illustration from http://people.sc.fsu.edu/~jburkardt/latex/asa_2011_graphs_homework/
Integer programming solves the Labs difficult discrete optimization problems.

Water, Road sensor placement, subway, building sensor management, Network interdiction, Scheduling quantum EC, Protein structure, Peptide docking, Meshing, Space-filling curves, Energy systems, Pantex planning, Vehicle routing, Conference schedule.

**Integer Program:** Minimize a linear cost function subject to linear inequality and integrality constraints.

\[
\begin{align*}
\text{minimize} & \quad c \cdot x \\
\text{subject to} & \quad A \cdot x \geq b \\
& \quad x \in \mathbb{Z}^n.
\end{align*}
\]
Recent focus on creating formulations of identified (tractible) problem structures.

Hardness arguments of modeling difficult structures.

Predict solution efficiency of a formulation.

Lockheed Martin Tech Refresh (Watson), improved formulation changed solution times from days to minutes.
Find minimum cost **2-edge connected** spanning subgraph.

**Protect** shipments **against single failure**.

**Doubled edges** are allowed and provided at a discount.
\[ \delta(S) := \{ e = \{i, j\} \in E : |S \cap e| = 1 \}, \]
\[ E(S) := \{ e = \{i, j\} \in E : |S \cap e| = 2 \} \quad \forall S \subset V, \]
\[ x(F) := \sum_{e \in F} x_e \quad \forall F \subset E. \]

A Classic 2-edge connected spanning subgraph problem.

\[ x_e \in \{0, 1, 2\} \text{ vars: Buy edge at price } c_e. \]

\[
\begin{align*}
\min & \quad c \cdot x \\
\text{subj to} & \quad x(\delta(S)) \geq 2 \quad \forall S \subset V, \\
& \quad 0 \leq x_e \leq 2 \quad \forall e \in E, \\
& \quad x_e \in \mathbb{Z} \quad \forall e \in E.
\end{align*}
\]

Drop integrality constraints to get LP relaxation.
Double-tree and Christofides heuristics

Select a minimum cost spanning tree $T = (V, E^T)$.  

The edge incidence vector: 
$\chi^T_e = 1$ iff $e \in E^T$ else $\chi^T_e = 0$.

Double each $e \in E^T$: $2\chi^T$ is the multi-edge incidence vector (has 2s) of our 2-edge connected graph.

Take the set $T^{odd}$ of odd degree nodes of $T$.

A $T^{odd}$-join is a graph $M = (V, E^M)$ such that the degree of $v \in V$ is odd iff $v \in T^{odd}$.

Select a minimum cost $T^{odd}$-join $M$.

$\chi^T + \chi^M$ is the multi-edge incidence vector of a connected, Eulerian, hence 2-edge connected graph.
Double-tree approximation

Let $x^*$ be optimal for LP relaxation.

$$x^*(\delta(S)) \geq 2 \quad \forall S \subset V$$ implies that $x^*$ satisfies the partition inequalities for spanning trees.

Since $x^*$ satisfies the partition inequalities, $x^*$ dominates a convex combination of incidence vectors of spanning trees:

$$x^* \geq \sum_i \lambda_i \chi^{T,i}, \quad (\sum_i \lambda_i = 1).$$

Each tree can be doubled to get a 2-edge connected graph:

$$2x^* \geq \sum_i \lambda_i (2\chi^{T,i}).$$

By averaging argument, one 2-edge connected $2\chi^{T,i}$ costs at most that of $2x^*$. 
**Christofides approximation**

\( x^* \) dominates a convex combination of tree vectors:

\[
x^* \geq \sum_i \lambda_i \chi_{T,i}^\ast.
\]

Let \( T_i \) be set of odd degree nodes for tree \( i \).

\[
\frac{1}{2} x^* (\delta(S)) \geq 1 \quad \forall S \subset V \text{ implies that } \frac{1}{2} x^* \text{ satisfies the } T_i \text{-join inequalities for each } i.
\]

Since \( \frac{1}{2} x^* \) satisfies the \( T_i \)-join inequalities, \( \frac{1}{2} x^* \) dominates a convex combination of \( T_i \)-join vectors:

\[
\frac{1}{2} x^* \geq \sum_j \mu_{ij} \chi_{M,ij}, \quad (\sum_j \mu_{ij} = 1).
\]

For each \( i, j \), \( \chi_{T,i}^T + \chi_{M,ij}^M \) is 2-edge connected.

\[
\frac{3}{2} x^* \geq \sum_i \sum_j \lambda_i \mu_{ij} (\chi_{T,i}^T + \chi_{M,ij}^M).
\]

By averaging argument, one 2-edge connected \( \chi_{T,i}^T + \chi_{M,ij}^M \) costs at most that of \( \frac{3}{2} x^* \).
Our new 2-edge connected problem

\[ x_e \in \{0, 1, 2\} \text{ vars: Buy each edge at price } c_e, \]
\[ y_e \in \{0, 1\} \text{ Buy doubled edge at a discount,} \]
\[ x \oplus y \in \mathbb{R}^E \times \mathbb{R}^E, \]
\[ c \oplus c' \in \mathbb{R}^E \times \mathbb{R}^E (c'_e \leq 2c_e). \]
\[ \min (c \oplus c') \cdot (x \oplus y) \]
\[ \text{subj to} \]
\[ x(\delta(S)) + 2y(\delta(S)) \geq 2 \quad \forall S \subseteq V \]
\[ 0 \leq x_e \leq 2, 0 \leq y_e \leq 1 \quad \forall e \in E \]
\[ x_e, y_e \in \mathbb{Z}. \]

Drop integrality constraints to get LP relaxation.

**Integrality gap of 2:** \[ y_e^* = \frac{1}{2} \] for edges of a Hamilton \((n \text{ edge})\) cycle. But optimal integer solution is \[ y_e^{opt} = 1 \] for all but one edge of cycle.
A better LP relaxation

Idea: $x_e + y_e$ dominates a spanning tree vector, denoted by $z_e$. That is, $x + y$ has enough mass ($n - 1$ edges) to contain a spanning tree, and the tree ($z$) has acyclic structure.

Add $x_e + y_e \geq z_e \ z(E(V)) = n - 1,$
\[
\forall S \subset V \ z(E(S)) \leq |S| - 1.
\]

Now $y_e^* = \frac{1}{2}$ on a Hamilton cycle no longer feasible.

Worst gap seen is now $\frac{3}{2}$ when horizontal edges of a square have $x_e^* = 1$ and a cost of 1 and vertical edges of that square have $y_e^* = \frac{1}{2}$ and a cost of 2.
Let $x^* \oplus y^*$ be an optimal extreme point solution to our LP.

To keep things simple, assume $x_e^* = 0$ or $y_e^* = 0$ for each $e \in E$.

$x_e^* + y_e^* \geq z_e^*$ and the spanning tree constraints on $z^*$ imply that $x^* + y^*$ dominates a convex combination of incidence vectors $\chi^{T,i}$ of spanning trees $x^* + y^* \geq \sum_i \lambda_i \chi^{T,i}$.

For each spanning tree, break up its set of edges into a set of $x$-edges and a set of $y$-edges. Then its incidence vector $\chi^{T,i}$ is broken up into incidence vectors $\chi^{T,x,i}$ and $\chi^{T,y,i}$.

So, $\chi^{T,i} = \chi^{T,x,i} + \chi^{T,y,i}$. Thus, $x^* + y^* \geq \sum_i \lambda_i (\chi^{T,x,i} + \chi^{T,y,i})$.

Finally, $x^* \oplus y^* \geq \sum_i \lambda_i (\chi^{T,x,i} \oplus \chi^{T,y,i})$. 
Double-tree approximation

Let $x^* \oplus y^*$ be optimal for LP relaxation.

$x^*_e + y^*_e \geq z^*_e$ and the constraints on $z^*$ imply that $x^* \oplus y^*$ dominates a convex combination of incidence vectors of spanning trees in $x \oplus y$ variable space:

$$x^* \oplus y^* \geq \sum_i \lambda_i (\chi^{T,x,i} \oplus \chi^{T,y,i}).$$

The $x$-part of each tree can be doubled to get a 2-edge connected graph:

$$2x^* \oplus y^* \geq \sum_i \lambda_i (2\chi^{T,x,i} \oplus \chi^{T,y,i}).$$

By averaging argument, one 2-edge connected $2\chi^{T,x,i} \oplus \chi^{T,y,i}$ costs at most that of $2x^* \oplus y^*$. 
Christofides approximation

\( x^* \oplus y^* \) dominates a convex combination of tree vectors: 
\[
 x^* \oplus y^* \geq \sum_i \lambda_i (\chi^{T,x,i} \oplus \chi^{T,y,i}).
\]

Let \( T_i \) be set of odd degree nodes for tree \( i \).
\[
\frac{1}{2}(x^*(\delta(S)) + y^*(\delta(S))) \geq 1 \quad \forall S \subset V
\]
implies that \( \frac{1}{2}x^* + y^* \) satisfies \( T_i \)-join inequalities for each \( i \).

Since \( \frac{1}{2}x^* + y^* \) satisfies the \( T_i \)-join inequalities, 
\[
\frac{1}{2}x^* \oplus y^* \geq \sum_j \mu_{ij} (\chi^{M,x,ij} \oplus \chi^{M,y,ij}).
\]

For each \( i,j \), 
\[
(\chi^{T,x,i} + \chi^{M,x,ij}) \oplus (\chi^{T,y,i} + \chi^{M,y,ij})
\]
is 2-edge connected.

\[
\frac{3}{2}x^* \oplus 2y^* \geq \sum_i \sum_j \lambda_i \mu_{ij} (\chi^{T,x,i} + \chi^{M,x,ij}) \oplus (\chi^{T,y,i} + \chi^{M,y,ij}).
\]

By averaging argument, one 2-edge connected 
\[
(\chi^{T,x,i} + \chi^{M,x,ij}) \oplus (\chi^{T,y,i} + \chi^{M,y,ij})
\]
costs at most that of \( \frac{3}{2}x^* \oplus 2y^* \).
The 5/3 approximation

From the Double-tree approximation, $2x^* \oplus y^*$ dominates a convex combination of 2-edge connected graphs $G^1_i$.

From the Christofides approximation, $\frac{3}{2}x^* \oplus 2y^*$ dominates a convex combination of 2-edge connected graphs $G^2_i$.

We can combine these as follows:

$$\frac{1}{3} (2x^* \oplus y^* \geq \sum_{i} \lambda_i G^1_i) + \frac{2}{3} (\frac{3}{2}x^* \oplus 2y^* \geq \sum_{i} \lambda_i G^2_i)$$

$$\frac{5}{3}x^* \oplus \frac{5}{3}y^* \geq \sum_{i} \lambda_i G_i.$$ 

The $\frac{5}{3}$ approximation and integrality gap follows since one of the $G_i$'s cost at most that of $\frac{5}{3}(x^* \oplus y^*)$ by our averaging argument.