Efficient Solution Algorithms for Partial Differential Equations with Random Coefficients

Navier-Stokes Solvers and Generalizations for Reacting Flow Problems

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Abstract

We consider new computational methods for solving partial differential equations (PDEs) when components of the problem, for example, the diffusion coefficient in the diffusion equation $-\nabla a \cdot \nabla u = f$, are random fields. A common source of such problems is in models of diffusive flow where permeabilities of transporting media are unknown. In recent years, several computational techniques have been developed for such models that offer potential for improved efficiencies compared with traditional Monte-Carlo methods. These include stochastic Galerkin methods, which use an augmented weak formulation of the PDE derived from averaging with respect to expected value, and stochastic collocation methods, which can be viewed as specialized sampling methods using a set of samples relatively small in cardinality that captures the character of the solution space. Both approaches are known to display convergence rates exponential with respect to polynomial degrees associated with discretization of the stochastic component of the problem.

Our aim here is threefold. First, we give an overview of the relative advantages of these two methods and explore their performance. Galerkin methods have the disadvantage of requiring solutions of coupled systems of equations many times larger than those induced by spatial discretization. Collocation methods require solutions only of spatially discrete problems, but the number of such solves can be significantly larger than the discrete stochastic component of Galerkin methods that achieve comparable accuracy. We show that for problems in which the dependence on uncertain parameters is linear, as when the diffusion coefficient is given as a finite-term approximation to its Karhunen-Loève expansion, the Galerkin systems can be solved efficiently by multigrid methods so that the overall cost of solution is significantly lower than for collocation.

In addition, we discuss refinements of both these methodologies. A commonly used model takes the diffusion coefficient to be of *log-normal* structure, $a = \exp(c)$, where c is a normally distributed random field. This ensures well-posedness of the problem but the dependence on uncertain parameters is nonlinear, and the utility of Galerkin methods is diminished. This difficulty can be alleviated by reformulating the problem in convection-diffusion form, as

$$-\nabla^2 u + (\nabla a) \cdot (\nabla u) = e^{-a} f.$$

We show that efficient solvers for convection-diffusion problems, also based on multigrid, can be used to recover the rapid convergence for Galerkin systems.

Finally, computation of statistical properties of solutions such as moments requires explicit knowledge of the joint density function associated with the (finite number of) random variables that determine the uncertain coefficients. This is often not available, and instead, computation is enabled under a simplifying and possibly false assumption that the joint density is simply the product of the marginal densities. We show that this assumption can be eliminated using techniques of kernel density estimation, and this approach can be used to develop adaptive collocation methods that are as effective as Monte-Carlo simulation and significantly less costly.