A fourth order accurate finite difference scheme for the elastic wave equation in second order formulation[1]

B. Sjögreen and N.A.Petersson

Lawrence Livermore National Laboratory

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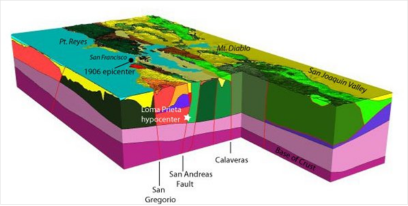
Computational seismology

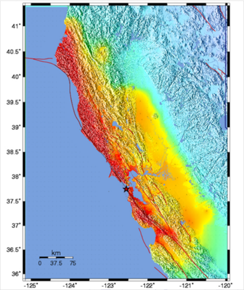
$$\rho\, \mathbf{u}_{tt} = div\sigma + \mathbf{f}$$

- $\mathbf{u} = \mathbf{u}(x, y, z, t)$ displacement vector ($\mathbf{u} = (u \;\; v \;\; w)$).
- $\mathbf{f} = \mathbf{f}(x, y, z, t)$ forcing = earthquake model
- stress tensor:

$$\sigma = \begin{pmatrix} (2\mu+\lambda)u_x & \mu(u_y+v_x) & \mu(u_z+w_x) \\ \mu(u_y+v_x) & (2\mu+\lambda)v_y & \mu(v_z+w_y) \\ \mu(u_z+w_x) & \mu(v_z+w_y) & (2\mu+\lambda)w_z \end{pmatrix}$$

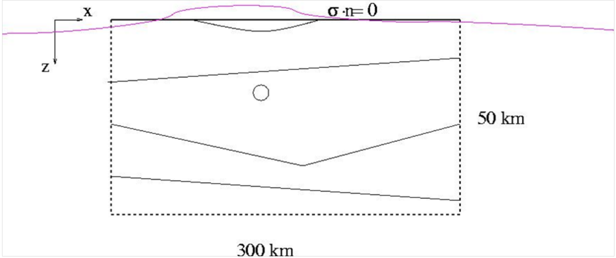
- $\rho = \rho(x, y, z)$, $\mu = \mu(x, y, z)$, $\lambda = \lambda(x, y, z)$ mtrl. prop.





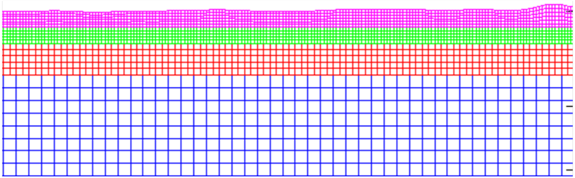
Domain and wave types

Computational domain



- Traction free boundary condition at surface.
- Pressure wave with speed $c_p = \sqrt{(2\mu + \lambda)/\rho}$.
- Shear wave with speed $c_s = \sqrt{\mu/\rho}$.
- Wave speed ratio $c_p/c_s > \sqrt{2}$.
- Rayleigh waves on surface, slower than P- and S-waves.

Topography handled by curvilinear grid



Grid refinement for depth varying wave speeds.

Resolution requirements

$$h = \frac{\min c_s}{Pf}$$

- Grid spacing h
- Points per shortest wavelength P
- Highest frequency f
- Material shear wave speed c_s

Typical values: $f = 10$ Hz, $c_s = 300$ m/s, $P = 15$ (second order), $P = 7$ (fourth order), gives

$$h = 2m \;\; \text{(2nd order)} \quad h = 4.29m \;\; \text{(4th order)}$$

Domain size 200 km $\rightarrow$ 100,000 pts/dimension (2nd) 46,620 (4th)

Objective

This work (new): 4th order accurate energy conserving method.

Previous work: 2nd order accurate energy conserving method. Extensions to

- Curvilinear grids
- Far field boundaries
- Mesh refinement
- Viscoelastic model

Energy conserving methods for the elastic wave equation

E^n discrete energy at t_n, integral over space, conserved when $\mathbf{f} = \mathbf{0}$

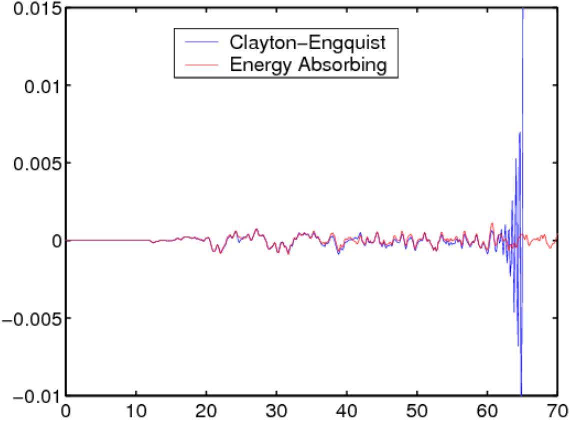
$$E^n = E^{n-1} = \ldots = E^0.$$

Compatibility with norm, $c_1 ||u^n||_h \le E^n \le c_2 ||u^n||_h$ gives stability,

$$||u^n||_h \le E^n/c_1 = \ldots = E^0/c_1 \le c_2/c_1 ||u^0||_h$$

- Stability for inhomogeneous material, real b.c., any c_p/c_s.
- Stable for long time integration
- Dissipation free
- Robust code, no numerical parameters to tune, but must be careful to not introduce unresolved frequencies

Energy estimate gives long time stability



Standard stability gives convergence on $0 < t < T$ with T fixed.

Example: Wave equation with mixed derivative term

$$u_{tt} = (2au_x + au_y)_x + (au_x + 2au_y)_y, \quad (x, y) \in [0, 1]^2, \;\; t > 0$$

$a = a(x, y) > 0$ variable coefficient. Boundary conditions:

$$\begin{aligned} u &= 0 && \text{at } \; x = 0 \\ 2u_x + u_y &= 0 && \text{at } \; x = 1 \\ u(x, y, t) &= u(x, y+1, t) && \text{(periodic in } \; y) \end{aligned}$$

Energy estimate

$$\frac{1}{2}\frac{d}{dt}\left(||u_t||^2 + (u_x, au_x) + (u_x + u_y, a(u_x + u_y)) + (u_y, au_y)\right) = 0$$

(Note: Non-negative terms give L^2 estimate)
Derived by partial integration:

$$\frac{1}{2}\frac{d}{dt}||u_t||^2 = (u_t, u_{tt}) = \ldots =$$
$$-\frac{1}{2}\frac{d}{dt}\left((u_x, 2au_x) + (u_x, au_y) + (u_y, au_x) + (u_y, 2au_y)\right) + B.T$$

Energy terms: $(u_x, au_x) + (u_x + u_y, a(u_x + u_y)) + (u_y, au_y)$
$B.T. = u_t a(2u_x + u_y)|_{x=1} - u_t a(2u_x + u_y)|_{x=0}$ zero by b.c.

Discretization

Cartesian grid with constant spacing h.
Centered finite difference operators

$$\partial u(x_i)/\partial x \to D_0 u_i, \quad i = 1, \dots, N$$

satisfying summation-by-parts

$$(u, D_0 v)_h = -(D_0 u, v)_h + u_N v_N - u_1 v_1$$

in a discrete, weighted, scalar product $(u, v)_h$. Further notation:

$$D_+ u_i = (u_{i+1} - u_i)/h, \qquad D_- u_i = (u_i - u_{i-1})/h.$$

In two dimensions: $D_0^{(x)} u_{i,j}$ and $D_0^{(y)} u_{i,j}$.

Discretization

$(au_y)_x \approx D_0^{(x)}(aD_0^{(y)}u)$ and $(au_x)_x \approx D_0^{(x)}(aD_0^{(x)}u)$ same energy estimate as for PDE possible, but

- Energy not positive definite, norm estimate not possible.
- Boundary condition $2u_x + u_y = 0$ implicit.

Second order method $(au_x)_x \approx D_+(a_{j-1/2}D_-u_j)$, where Energy estimate based on

$$D_+(a_{j-1/2}D_-u_j) = D_0(a_jD_0u_j) - \frac{h^2}{4}D_+D_-(a_jD_+D_-u_j),$$

Square completion with x-y terms Keeps energy pos. def.

Use of ghost points, gives explicit discrete b.c. with no boundary modification of D_+D_-.

Fourth order accurate operator

$$(au_x)_x \approx G(a,u)_j = D_0(a_j D_0 u_j) + \frac{h^4}{18} D_+ D_- D_+ (a_{j-1/2} D_- D_+ D_- u_j)$$
$$- \frac{h^6}{144} (D_+ D_-)^2 (a_j (D_+ D_-)^2 u_j) + \text{boundary modifications}$$

- G is five point wide operator away from the boundary.
- D_0 SBP operator of order $4/2$, needed for xy-derivatives.
- G also order $4/2$. Boundary modified at $j = 1, \ldots, 6$.
- B.T.=0 in SBP is 4th order accurate b.c. $\rightarrow$ 4th order error.
- Boundary modification of $(D_+ D_-)^3$ gives first order errors that can be made to cancel first order errors of $D_0(aD_0 u)$.
- Can expand $G(a,u)_j = \sum_{m=1}^{8} \sum_{k=1}^{8} \beta_{j,k,m} a_k u_m$, $j = 1, \ldots, 6$. Coefficient tensor β with 129 non-zero elements out of 384.
- G uses ghost points, D_0 does not.

4th order P-C time discretization gives energy conservation

Can prove time discrete energy conservation:

$$E^{n+1/2} = E^{n-1/2}.$$

Method stable (energy positive) for $CFL < 1.3$. No stiffness for high order.

Numerical examples

Elastic wave equation, 2D

$$
\begin{aligned}
\rho u_{tt} &= ((2\mu+\lambda)u_x)_x + (\lambda v_y)_x + (\mu v_x)_y + (\mu u_y)_y \\
\rho v_{tt} &= (\mu v_x)_x + (\mu u_y)_x + (\lambda u_x)_y + ((2\mu+\lambda)v_y)_y
\end{aligned}
$$

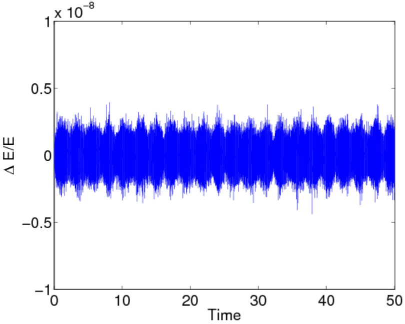
$0 < x < L_x$, $0 < y < L_y$, $t > 0$.
Initial data: $u(x, y, 0)$ and $u_t(x, y, 0)$ given.
Boundary data: y-periodic, with Dirichlet b.c. on $x = L_x$ and

$$
\begin{aligned}
(2\mu+\lambda)u_x + \lambda v_y &= 0 \qquad x = 0 \\
\mu(v_x + u_y) &= 0 \qquad x = 0
\end{aligned}
$$

Energy test with random material

$$\rho(x,y) = 4 + \theta \quad \mu(x,y) = 2 + \theta \quad \lambda(x,y) = 2(r^2 - 2) + \theta$$

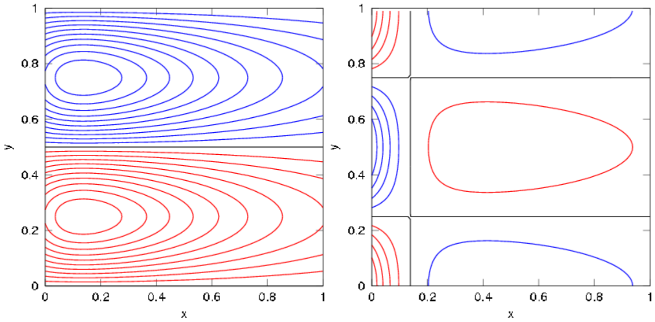
Random variable $\theta \in [0,1]$. Approximate wave speed ratio $r = c_p/c_s$. Initial data also random numbers.



Energy change per time step. Total $> 220,000$ steps.
c_p/c_s arbitrarily large.

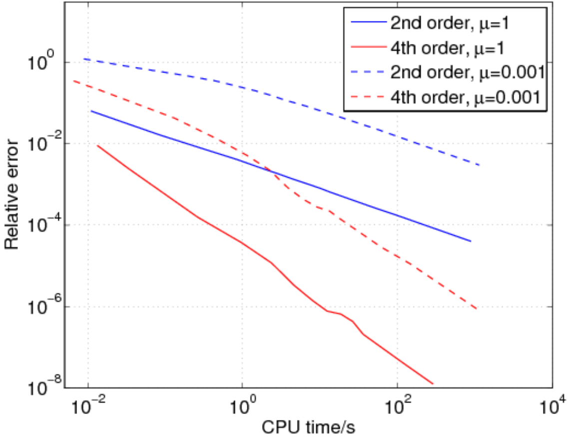
Rayleigh waves

Surface waves at $x = 0$, solutions $\mathbf{u}_s$ traveling wave in y and decaying as e^{-ax} into the domain.



μ, λ, and ρ constant.

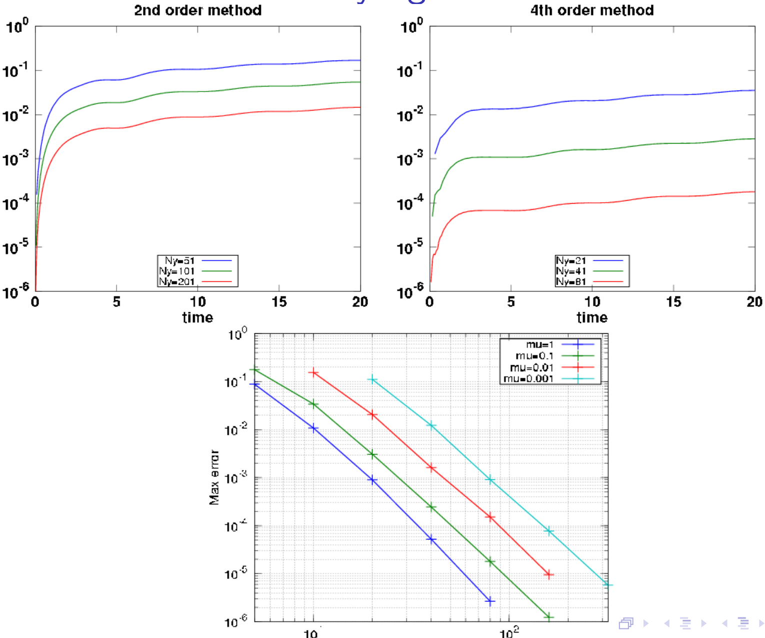
34 seconds vs. 54 hours CPU time



Error vs. CPU time

$\mu = 0.001$, error 10^{-4} need 34 seconds with 4th order scheme, 54 hours with 2nd order scheme.

Rayleigh waves



Summary and future directions

- 4th order accurate non-dissipative difference scheme, L^2 norm stable with heterogeneous material and boundary conditions.
- 4th order in both space and time.
- Significant savings in computational resources.
- High order second derivative approximation of $(\mu(x)u_x)_x$, with norm stable boundary closure, useful in other applications.

- To be implemented into the 3D WPP solver.
- To be used in new solver for source and material inversion, using adjoint wave propagation.

Reference

[1] B. Sjögreen and N.A.Petersson, *A fourth order finite difference scheme for the elastic wave equation in second order formulation*, Lawrence Livermore National Laboratory, LLNL-JRNL-483427, (to appear in J.Scient.Comput.).