# Functions



## Partial functions

A partial function from $X$ to $Y$ is a relation that maps each element of $X$ to at most one element of $Y$.

$$\begin{aligned}
&X \rightarrowtail\!\!\!\!\!\!\!+\; Y \;==\\
&\quad \{ f : X \leftrightarrow Y \mid\\
&\qquad \forall x : X;\ y_1, y_2 : Y \bullet\\
&\qquad\quad x \mapsto y_1 \in f \land x \mapsto y_2 \in f \Rightarrow y_1 = y_2 \}
\end{aligned}$$



## Example

$$\begin{array}{|l}
where_is : Person \nrightarrow Location \\
\hline
where_is = \{otto \mapsto lobby, peter \mapsto meeting, \\
\qquad\qquad quentin \mapsto meeting, rachel \mapsto meeting\}
\end{array}$$



## Total functions

If each element of the source set is related to some element of the target, then the function is said to be total.

$$X \rightarrow Y \;==\; \{f : X \nrightarrow Y \mid \operatorname{dom} f = X\}$$



## Example

$$\begin{array}{|l}
double : \mathbb{N} \leftrightarrow \mathbb{N} \\
\hline
\forall m, n : \mathbb{N} \bullet m \mapsto n \in double \Leftrightarrow m + m = n
\end{array}$$



## Application

$$\frac{\exists_1 p : f \bullet p.1 = a \qquad a \mapsto b \in f}{b = f\ a}\ [\text{app}-\text{intro}]$$

provided that $b$ does not appear free in $a$

## Conversely

$$\frac{\exists_1\, p : f \bullet p.1 = a \qquad b = f\ a}{a \mapsto b \in f}\ [\text{app}-\text{elim}]$$

provided that $b$ does not appear free in $a$



## Examples

- $\mathit{where\_is}\ \mathit{rachel}\ =\ \mathit{meeting}$
- $\mathit{double}\ 7\ =\ 14$
- $\mathit{double}\ {-1}\ =\ 2$ ?



## Lambda notation

$\lambda\ \mathit{declaration} \mid \mathit{constraint} \bullet \mathit{result}$

The source type of such a function is the Cartesian product of the types in the declaration.



## Example

$$\begin{array}{|l}
\mathit{double} : \mathbb{N} \rightarrow \mathbb{N} \\
\hline
\mathit{double} = \lambda\, m : \mathbb{N} \bullet m + m
\end{array}$$



## Toolkit functions

The operators defined upon relations, such as ‘dom’ and ‘$\lhd$’, are all functions.



## Example

$$\begin{array}{|l}
[X, Y] \\
\mathrm{dom} : (X \leftrightarrow Y) \rightarrow \mathbb{P}\, X \\
\mathrm{ran} : (X \leftrightarrow Y) \rightarrow \mathbb{P}\, Y \\
\hline
\forall\, R : X \leftrightarrow Y \bullet \\
\quad \mathrm{dom}\, R = \{\, x : X \mid \exists\, y : Y \bullet x \mapsto y \in R \,\} \\
\quad \mathrm{ran}\, R = \{\, y : Y \mid \exists\, x : X \bullet x \mapsto y \in R \,\}
\end{array}$$

## Fixity

For convenience, we may decide that function names are to be used as prefix, infix, or suffix symbols.

In the definition of infix and suffix symbols, we use underscores to indicate the placement of arguments.



## Example

$$
\begin{array}{l}
[X, Y] \\
\_ \lhd \_ : \mathbb{P} X \times (X \leftrightarrow Y) \rightarrow (X \leftrightarrow Y) \\
\_ \rhd \_ : (X \leftrightarrow Y) \times \mathbb{P} Y \rightarrow (X \leftrightarrow Y) \\
\hline
\forall R : X \leftrightarrow Y;\ A : \mathbb{P} X;\ B : \mathbb{P} Y \bullet \\
\quad A \lhd R = \\
\qquad \{ x : X;\ y : Y \mid x \in A \land x \mapsto y \in R \bullet x \mapsto y \} \\
\quad R \rhd B = \\
\qquad \{ x : X;\ y : Y \mid y \in B \land x \mapsto y \in R \bullet x \mapsto y \}
\end{array}
$$



## Example

$$
\begin{array}{l}
[X, Y] \\
\_^{\sim} : (X \leftrightarrow Y) \rightarrow (Y \leftrightarrow X) \\
\hline
\forall R : X \leftrightarrow Y \bullet \\
\quad R^{\sim} = \{ x : X;\ y : Y \mid x \mapsto y \in R \bullet y \mapsto x \}
\end{array}
$$



## Question

Suppose that $f$ and $g$ are functions. Is it necessarily the case that $f \cup g$ is a function?



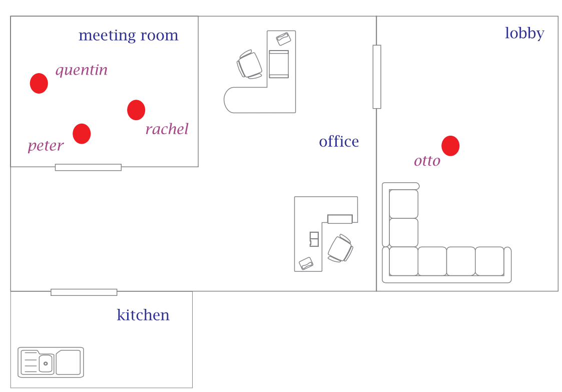
## Overriding

If $f$ and $g$ are functions of the same type, then $f \oplus g$ is a function that agrees with $f$ everywhere outside the domain of $g$; but agrees with $g$ where $g$ is defined:

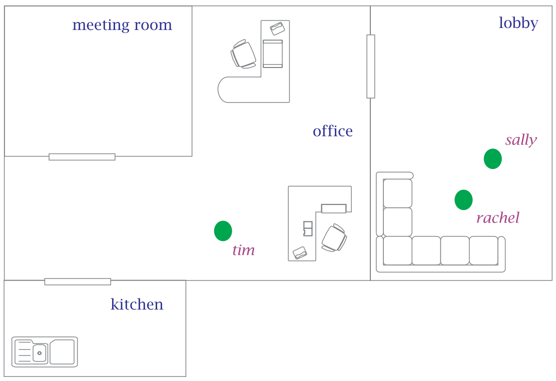
$$
\begin{array}{l}
[X, Y] \\
\_ \oplus \_ : (X \leftrightarrow Y) \times (X \leftrightarrow Y) \rightarrow (X \leftrightarrow Y) \\
\hline
\forall f, g : X \leftrightarrow Y \bullet \\
\quad f \oplus g = (\operatorname{dom} g \ntriangleleft f) \cup g
\end{array}
$$



## Original

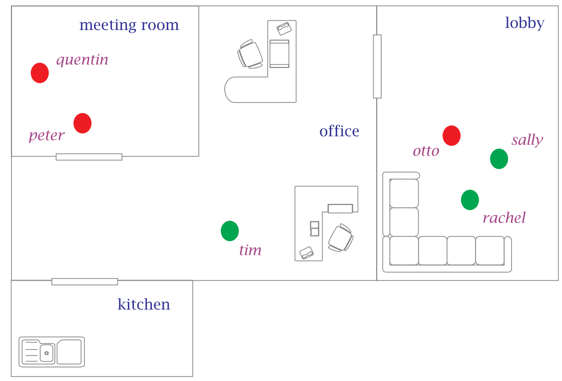


## Update






## Override






## Properties of functions

| | |
|---|---|
| ⤔ | partial, injective functions |
| $\rightarrowtail$ | total, injective functions |
| ⤀ | partial, surjective functions |
| $\twoheadrightarrow$ | total, surjective functions |
| ⤗ | partial, bijective functions |
| ⤖ | total, bijective functions |



## Injections, surjections, and bijections

If each element of the domain is mapped to a different element of the target, then the function is injective. (*1 to 1*)

If the range of the function is the whole of the target, then the function is surjective. (*onto*)

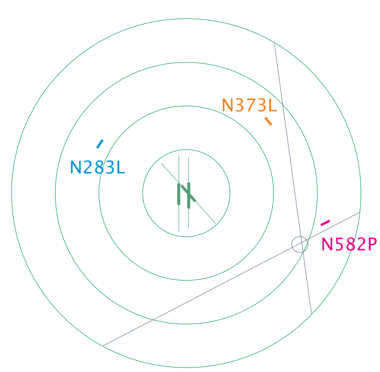
A function which is both injective and surjective is said to be bijective. (*1 to 1 correspondence*)



## Example

*locate* : *CallSign* ⤔ *Position*

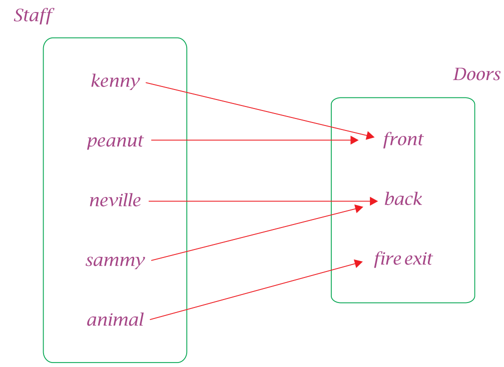




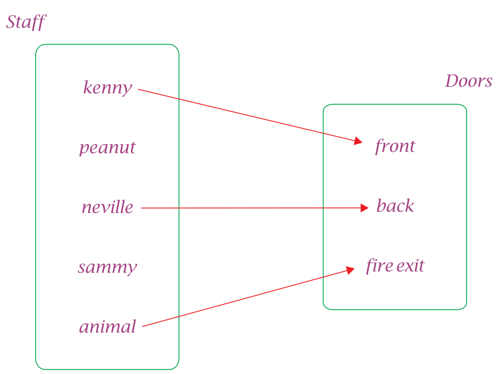
## Example

$station : Staff \twoheadrightarrow Door$







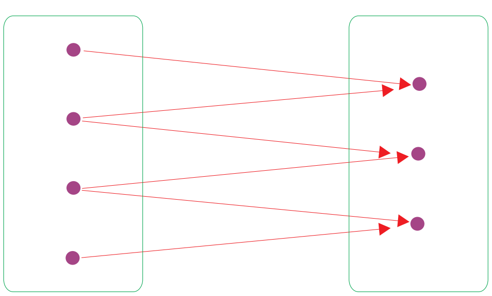



## Functionality test

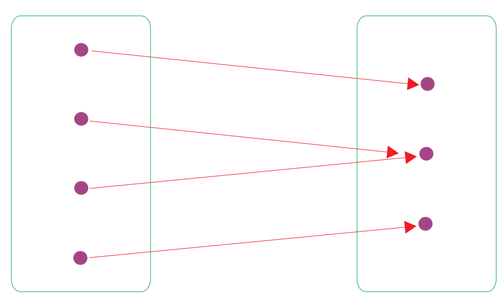
Following this slide there are five graphs. Which of the relations illustrated are

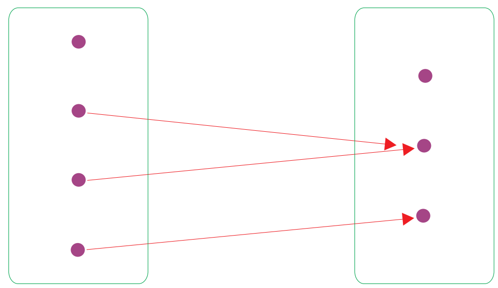
- functional?
- injective?
- surjective?
- bijective?



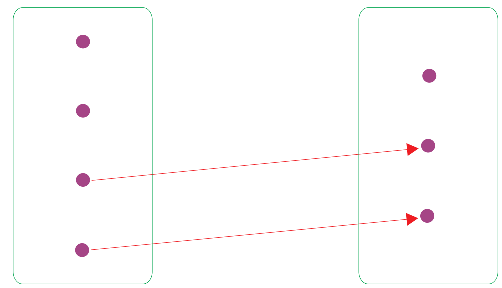




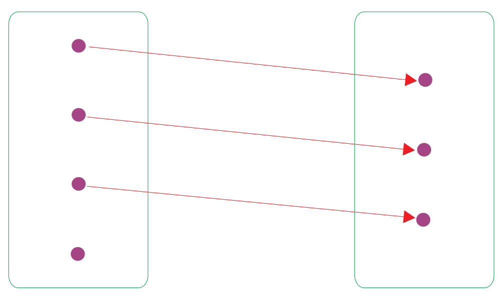














## Finite sets

A finite set is one whose elements are countable up to some natural number $n$: that is, a set that may be seen as the range of a total bijection from $1, 2, \ldots, n$.

$$\mathbb{F}\, X \;==\; \{\, s : \mathbb{P}\, X \mid \exists\, n : \mathbb{N} \bullet \exists\, f : 1\,..\,n \rightarrowtail\!\!\!\!\twoheadrightarrow s \bullet \mathit{true} \,\}$$

where

$$\_\,..\,\_ : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{P}\,\mathbb{N}$$

$$\forall\, m, n : \mathbb{N} \bullet m\,..\,n = \{ i : \mathbb{N} \mid m \leq i \leq n \}$$



## Question

Which of the following sets are finite?

- *Oceans*
- *Jose Felicianos*
- *Rockallers*
- *Primes*



## Hash, or cardinality

$[X]$

$$\# : \mathbb{F}\, X \rightarrow \mathbb{N}$$

$$\forall\, s : \mathbb{F}\, X;\; n : \mathbb{N} \bullet$$
$$\quad n = \# s \Leftrightarrow \exists\, f : (1\,..\,n) \rightarrowtail\!\!\!\!\twoheadrightarrow s \bullet \mathit{true}$$

## Finite functions

- finite functions:

$$A \mathrel{+\!\!\!\!\twoheadrightarrow} B \;==\; \{ f : A \mathrel{+\!\!\!\!\rightarrow} B \mid \operatorname{dom} f \in \mathbb{F} A \}$$

- finite injections:

$$A \mathrel{\rightarrowtail\!\!\!\!\!+\!\!\!\!\rightarrow} B \;==\; A \mathrel{+\!\!\!\!\twoheadrightarrow} B \cap A \mathrel{\rightarrowtail\!\!\!\!\!+\!\!\!\!\rightarrow} B$$



## Summary

- special relations
- partial, total
- application
- $\lambda$ notation
- overriding
- injective, surjective, bijective
- finite functions